Differential Geometric Aspects in Image Processing

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Problem C1.1

i) Strictly convex $\Rightarrow$ strictly monotone ν:
Let c be strictly convex, then ν is monotone. Assume there exists $s_1 < s_2$ with $\nu(c(s_1)) = \nu(c(s_2))$. Then $\nu(c(s)) = \nu(c(s_1))$, for $s_1 \leq s \leq s_2$ and c would be a segment between s_1 and s_2 contradicting the strict convexity.

ii) Strictly monotone $\nu \Rightarrow$ strictly convex:
Let ν be strictly monotone, then c is convex. If c is not strictly convex it would contain a segment c and ν would be constant in that segment contradicting the strict monotonicity.

Problem C1.2

i) Let $c(p) = (x(p), y(p)) \in \mathbb{R}^2$ and $\tilde{c}(p) = Ac(p) + b$ with $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$ and $\det A = 1$. Then

$$\tilde{c}_p = Ac_p(p), \quad \tilde{c}_{pp}(p) = Ac_{pp}(p)$$

and

$$\begin{aligned}
\tilde{g} &= \det(\tilde{c}_p, \tilde{c}_{pp}) = \det(A\tilde{c}_p, A\tilde{c}_{pp}) \\
&= (a_{11}a_{22}x_p y_{pp} + a_{12}a_{21}x_{pp}y_p) - a_{12}a12x_p y_{pp} - y_p x_{pp} a_{22}a_{11}) \\
&= (a_{11}a_{22} - a_{21}a_{12})x_p y_{pp} - (a_{11}a_{22} - a_{21}a_{12})y_p x_{pp} \\
&= det(A)(x_p y_{pp} - y_p x_{pp}) = \det(c_p, c_{pp}) = g
\end{aligned}$$

ii) We have

$$\begin{aligned}
\tilde{s}(p) &= \int_0^p (\det(\tilde{c}_p(\tau), \tilde{c}_{pp}(\tau)))^{1/3}\, d\tau = \int_0^p (\det(A\tilde{c}_p(\tau), A\tilde{c}_{pp}(\tau)))^{1/3}\, d\tau \\
&= (\det A)^{1/3} \int_0^p (\det(c_p(\tau), c_{pp}(\tau)))^{1/3}\, d\tau = (\det A)^{1/3} s(p).
\end{aligned}$$

Hence

$$\frac{d\tilde{s}}{dp}\frac{dp}{ds} = (\det A)^{1/3} \frac{dp}{ds}\frac{ds}{dp}.$$

Problem C1.3

i) Group: Its clearly a group since $Id, A^{-1}, AB \in GL(d, \mathbb{R})$ for all $A, B \in GL(d, \mathbb{R})$.

ii) Manifold: $GL(d, \mathbb{R})$ is a manifold with one chart ϕ obtained by representing the matrices as vectors in $\mathbb{R}^{d\times d}$.
Notice that the image of ϕ is indeed an open set of $\mathbb{R}^{d\times d}$: In fact $\det(\phi^{-1}(v)) : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a smooth function (a polynomial of the components of v) thus $\det(\phi^{-1}(v)) > 0$ implies $\det(\phi^{-1}(w)) > 0$ for any w in a neighbourhood of v.

iii) Differentiable product and inversion: The product of two matrices is clearly smooth (elements of the resulting matrix are given by polynomials).
The inverse of a matrix $A \in GL(d, \mathbb{R})$ can be computed for example as

$$(A^{-1})_{ij} = \frac{(-1)^{i+j} \det(M_{ji})}{\det A},$$

where M_{ij} the determinant of the matrix obtain by removing the i'th row and j'th column of A, The differentiability thus follows from ii) and the fact that $\det \neq 0$ in $GL(d, \mathbb{R})$.