Differential Geometric Aspects in Image Processing

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Problem C1.1

i)

$$
\begin{aligned}
\phi_1 : &= \{y > 0\} \to]-a, a[, \quad \phi_1(x, y) = x \\
\phi_2 : &= \{y < 0\} \to]-a, a[, \quad \phi_1(x, y) = x \\
\phi_3 : &= \{x > 0\} \to]-b, b[, \quad \phi_1(x, y) = y \\
\phi_4 : &= \{x < 0\} \to]-b, b[, \quad \phi_1(x, y) = y
\end{aligned}
$$

To compute $\phi_1 \circ \phi_3 :]0, b[\to]0, a[$ we use the fact that

$$x = \pm\sqrt{a^2 - a^2y^2/b^2}$$

and obtain

$$\phi_3^{-1}(y) = \left(|a|\sqrt{1 - \frac{y^2}{b^2}}, y \right).$$

Thus

$$\phi_1 \circ \phi_3^{-1}(y) = |a|\sqrt{1 - \frac{y^2}{b^2}}$$

The other change of coordinates are similar.

ii)

$$
\begin{aligned}
\phi_1(x, y) &= \frac{y}{1 + x} \quad \text{defined on } \mathbb{S}^1 - \{(-1, 0)\} \\
\phi_2(x, y) &= \frac{y}{1 - x} \quad \text{defined on } \mathbb{S}^1 - \{(1, 0)\}
\end{aligned}
$$

Problem C1.2

Parametrise the graph of f as $c(p) = (p, f(p))$, and apply

$$\kappa(p) = \frac{x_1' x_2'' - x_2' x_1''}{((x_1')^2 + (x_2')^2)^{\frac{3}{2}}} \tag{1}$$

with $(x_1(p), x_2(p)) = (p, f(p))$ to obtain

$$k(c(p)) = \frac{f''(p)}{|1 + f'(p)^2|^{3/2}}.$$

We also have that $\overrightarrow{t} = \frac{(1, f'(p))}{\sqrt{1+f'(p)^2}}$, thus $\overrightarrow{n} = \frac{(-f'(p), 1)}{\sqrt{1+f'(p)^2}}$ with $(\overrightarrow{t}, \overrightarrow{n})$ positive oriented. The center of the osculating circle $\gamma(p)$ is thus

$$\gamma(p) = c(p) + \frac{1}{\kappa(p)}\overrightarrow{n} = (p, f(p)) + \frac{|1 + f'(p)^2|^{3/2}}{f''(p)} \cdot \frac{(-f'(p), 1)}{\sqrt{1 + f'(p)^2}}.$$

ii) Write the ellipse as the curve $c : [0, 2\pi] \to \mathbb{R}^2$, given by $c(p) = (acos(p), bsin(p))$ and apply the curvature formula (1).

Problem C1.3

The curvature of a circle with radius r is $\frac{1}{r}$, thus

$$\frac{dr}{dt} = -\frac{1}{r},$$

multiply by $-r$ and integrate over t to obtain

$$\int_0^T r\frac{dr}{dt}dt = T.$$

Therefore

$$\frac{r^2(T)}{2} - \frac{r^2(0)}{2} = T$$

and

$$r(T) = \sqrt{r_0^2 - 2T} \quad 0 \le T \le \frac{r_0^2}{2}.$$