Differential Geometric Aspects in Image Processing

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Problem H3.1 (6 Points)

i) We have that

$$\frac{d}{dt} < v(t), w(t) >=< v'(t), w(t) > + < v(t), w'(t) > .$$

On the other hand we can write the orthogonal decompositions

$$v'(t) = v_n(t) + v_\theta(t), \qquad w'(t) = w_n(t) + w_\theta(t),$$

with v_θ, w_θ in the corresponding tangent space and v_n, w_n normal to the surface. Therefore

$$\begin{aligned}\frac{d}{dt} < v(t), w(t) > &=< v_n(t) + v_\theta(t), w(t) > + < v(t), w_n(t) + w_\theta(t) > \\ &=< v_\theta(t), w(t) > + < v(t), w_\theta(t) >= \left\langle \frac{Dv}{dt}, w(t) \right\rangle + \left\langle v(t), \frac{Dw}{dt} \right\rangle,\end{aligned}$$

since $< v, w_\theta >= 0$ and $< w, v_\theta >= 0$.

ii) The ones corresponding to $(a + r, 0, 0)$ and $(a - r, 0, 0)$ are geodesics. The one corresponding to $(a, 0, r)$ is not.

Problem H3.2 (6 Points)

i) Let $\overrightarrow{t} = \gamma'$, $\overrightarrow{n}$ denote the unit tangent and normal of γ. We have that

$$\frac{\partial \sigma}{\partial u} = \overrightarrow{b}(v), \quad \frac{\partial \sigma}{\partial v} = \overrightarrow{t} + u\overrightarrow{b}'(v),$$

thus

$$\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} = \overrightarrow{b} \times \overrightarrow{t} + u\overrightarrow{b}\overrightarrow{b}'(v) = \overrightarrow{b} \times \overrightarrow{t} \neq 0$$

ii) For $u = 0$, the normal to the surface $\overrightarrow{N}$ corresponds to $\overrightarrow{b} \times \overrightarrow{t}$, a vector parallel to $\overrightarrow{n}$. Therefor $\gamma'' = \overrightarrow{n}$ is parallel $\overrightarrow{N}$, hence γ is a geodesic.