Differential Geometric Aspects in Image Processing

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Problem H1.1 (4 Points)

Assume $\xi = (\xi_1, \xi_2)^\top$. The directional derivative w.r.t. ξ is given by

$$\xi_1 \partial_x + \xi_2 \partial_y.$$

This lead to

$$u_{\xi\xi} = \partial_\xi(\partial_\xi u) = \xi_1^2 u_{xx} + 2\xi_1\xi_2 u_{xy} + \xi_2^2 u_{yy} = \xi^\top H(u)\xi$$

where $H(u)$ is the Hessian of u. In our case, ξ is a unit vector in level line direction, i.e. perpendicular to the local image gradient,

$$\xi_1 = \frac{u_y}{u_x^2 + u_y^2}, \qquad \xi_2 = \frac{-u_x}{u_x^2 + u_y^2}$$

Consequently

$$\begin{aligned} u_{\xi\xi} =& \frac{1}{u_x^2 + u_y^2}(u_y, -u_x)H(u)(u_y, -u_x)^\top && (1) \\ =& \frac{1}{u_x^2 + u_y^2}(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}). && (2) \end{aligned}$$

Multiplying finally the expression by the factor we recognize the expression for the curvature (derived in Lecture 4).

Problem H1.2 (4 Points)

i) The positive level set corresponding to distance d can be geometrically constructed by drawing from each point p of the curve, a segment of length d in the outwards orthogonal direction to the tangent at p. The set of end points of this family of segments gives the level set. The convexity of the curve ensures that this construction works for any $d > 0$.

ii) Assume without loss of generality that $a > b$. The distance function is differentiable everywhere except for the set $\{(x, 0) : x \in [-\sqrt{a} + \sqrt{b}, \sqrt{a} - \sqrt{b}]\}$

Problem H1.3 (4 Points)

One can easily verify that given such a matrix A and a vector $v \in \mathbb{R}^n$ the product $\tilde{v} = Av$ is given by a vector whose components correspond to a permutation of the components of v (if $a_{ij} = 1$ then $\tilde{v}_i = v_j$).
This family of matrices can thus be identified with the set of permutations of n elements, which is a group.