Lecture 20

- Low Dimensional Manifold Model
- Harmonic Extensions on Point Clouds: Point Integral Method
- Beltrami Flow

Patch Based Methods

- Non local means (Buades et. al 2005)

$$NL[u] = \frac{1}{C(x)} \int_{\Omega} \exp^{\frac{-(G_a * |u(x-\cdot)-u(y-\cdot)|^2)(0)}{h^2}} u(y)\, dy,$$

with

$$C(x) = \int_{\Omega} \exp^{\frac{-(G_a * |u(x-\cdot)-u(y-\cdot)|^2)(0)}{h^2}} dy,$$

where G_a is a Gaussian

- many other nonlocal methods for denoising or exemplar based inpainting (Gilboa et. al. 2009, Criminisi et. al. 2003, Arias et. al. 2011)

Patches and Manifolds

- for a given signal consider

$$\mathcal{M} = \{p_x(g) \,:\, x \in [0,1]^d, g \in \Theta\}$$

  with Θ a signal ensemble gathering the typical data of interest, with $p_x(g) \in L^2([x - \frac{\delta}{2}, x + \frac{\delta}{2}]^2)$ a patch of g centered at x

- main assumption: for natural images the set of patches (Lee et. al. 2003, Carlsson et. al. 2008) are well approximated by low dimensional manifolds

- examples of explicit patch manifolds (G. Peyré 2009):
  - manifold of smooth variations: C^1 images, patches well approximated by affine functions
  - manifold of cartoon images
  - manifold of locally parallel textures

Manifold of Smooth Variations

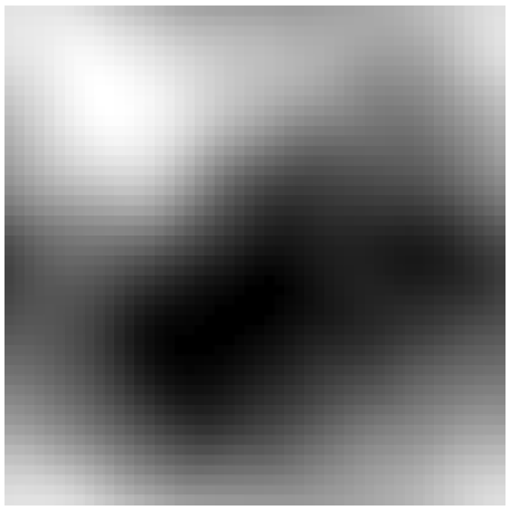
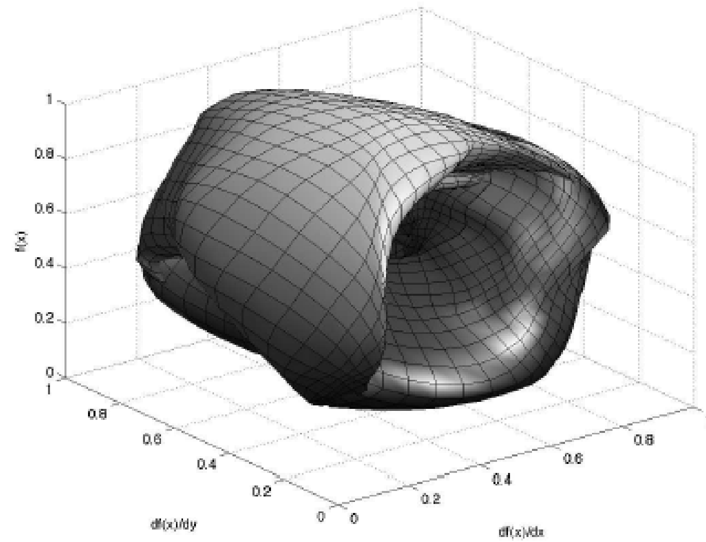


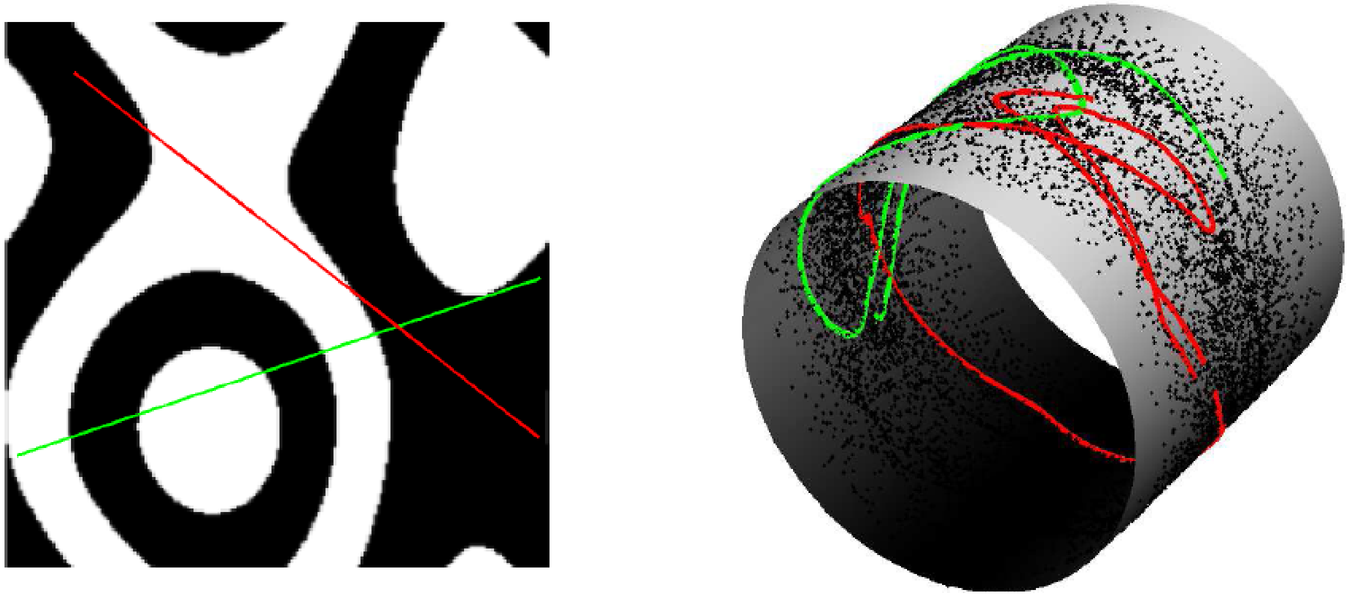
Image f



Surface $\tilde{c}_f$

Manifold of smooth images (G. Peyré 2009)

Manifold of Cartoon Images



Left to right: A cartoon image – A 3D representation of the edge manifold M (depicted in 3D as a cylinder). The two curves on the manifold corresponds to patches extracted along the two lines in the image (G. Peyré 2009)

locally parallel textures

Typical locally parallel texture (G. Peyré 2009)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Inverse Problems

- Many image processing problems can be formalized as the recovery of an image f from a set of noisy measurements Φf

$$y = \Phi f + \epsilon$$

- Φ typically accounts for some damage to the image, for instance, blurring, missing pixels, or downsampling
- In order to solve this ill-posed problem, one needs to have some prior knowledge of the image
- With the help of regularizations, many image processing problems are formulated as optimization problems, e.g.:

$$\operatorname*{argmin}_f R(f) + ||y - \Phi f||^2_{L^2}$$

Low Dimensional Manifold Regulariser

- assume that the patches of the image are well represented by a low dimensional manifold
- This leads to the following general for for the inverse problem (Osher et. al. 2017)

$$\underset{\mathcal{M}.f\in\mathbb{R}^{n\times m}}{\operatorname{argmin}} \operatorname{dim}(\mathcal{M}(f))(x)\,dx \quad \text{subject to} \quad y = \Phi f + \epsilon, \mathcal{P}(f) \subset \mathcal{M},$$

where $\mathcal{P}(f)$ are the patches of the image f.

Low Dimensional Manifold Regulariser

- **Lemma:** Let $\mathcal{M}$ be a submanifold isometrically embedded in $\mathbb{R}^d$. We have that

$$\dim(\mathcal{M}) = \sum_{j=1}^{d} ||\nabla_{\mathcal{M}} \alpha_j(\boldsymbol{x})||^2.$$

  where $\alpha_i(\boldsymbol{x}) = x_i$ for all $\boldsymbol{x} = (x_1, ..., x_d) \in \mathcal{M} \subset \mathbb{R}^d$.

- Therefore, the optimisation problem can be written (Osher et. al. 2017)

$$\operatorname*{argmin}_{\mathcal{M}, f \in \mathbb{R}^{n \times m}} \int_{\mathcal{M}} \sum_{j=1}^{d} ||\nabla_{\mathcal{M}} \alpha_j(\boldsymbol{x})||^2 d\boldsymbol{x} + \lambda ||y - \Phi f||^2 \quad \text{subject to} \quad \mathcal{P}(f) \subset \mathcal{M},$$

  where $\mathcal{P}(f)$ is the set of patches of f.

- Integrating the regularisation term over the whole manifold allows for possibly different dimensions at different parts.

Iterative Method

- Basic Structure:
  - With a guess of the manifold M^n and a guess of the image f^n satisfying $P(f^n) \subset M^n$, compute the coordinate functions $\alpha_i^{n+1}, i = 1, ..., d$, and f^{n+1} solving

$$(\alpha_1^{n+1}, ...\alpha_d^{n+1}, f^{n+1}) = \underset{\boldsymbol{\alpha}, f}{\operatorname{argmin}} \sum_{j=1}^{d} ||\nabla_{\mathcal{M}} \alpha_j(\boldsymbol{x})||^2 d\boldsymbol{x} + \lambda ||y - \Phi f||^2$$

subject to

$$\alpha_i(p_x(f^n)) = p_x^i(f),$$

where $p_x^i(f)$ is the ith element of patch $p_x(f)$

  - Update $\mathcal{M}$ by setting

$$\mathcal{M}^{n+1} = \{(\alpha_1^{n+1}(\boldsymbol{x}), ..., \alpha_d^{n+1}(\boldsymbol{x})) \, : \, \boldsymbol{x} \in \mathcal{M}^n\}$$

  - Repeat these two steps until convergence

Iterative Method

- the most difficult part is to solve the following type of optimization problem:

$$\min_{u\in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|^2_{L^2(\mathcal{M})} + \mu \sum_{\boldsymbol{y}\in P} |u(\boldsymbol{y}) - v(\boldsymbol{y})|^2$$

  where u can be any α_i, $\mathcal{M} = \mathcal{M}^n$, $P = \mathcal{P}(f^n)$, and $v(\boldsymbol{y})$ is a given function on P.

- the solution can be obtained by solving the PDE:

$$\Delta_{\mathcal{M}} u + \sum_{\boldsymbol{y}\in P} \delta(\boldsymbol{x} - \boldsymbol{y})(u(\boldsymbol{y}) - v(\boldsymbol{y})) = 0, \quad \boldsymbol{x} \in \mathcal{M}$$

$$\frac{\partial u}{\partial \boldsymbol{n}}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\mathcal{M}$$

  where $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$ and $\boldsymbol{n}$ is the outwards normal of ∂M. If M has no boundary, $\partial\mathcal{M} = \emptyset$.

- This PDE problem can be solved using the Point Integral Method

Harmonic Extension on Point Clouds

- Problem (Interpolation on a point cloud in high dimensional space):

  Let $P = \{\boldsymbol{p}_1, , \boldsymbol{p}_n\}$ be a set of points in $\mathbb{R}^d$. Let u be a function on P with known values only at $S \subset P$. From the given value on S, we want to recover the value of u on the whole data set P.

- To make the problem well-posed, we assume that the point cloud P sample a smooth manifold M embedded in $\mathbb{R}^d$

Harmonic Extension on Point Clouds

- Usually we do not known the manifold $\mathcal{M}$. Assume instead that we known weights w measuring the vicinity of points of the point cloud
- one option is to use the graph Laplacian:

$$\sum_{\boldsymbol{y}\in P}(w(\boldsymbol{x},\boldsymbol{y})+w(\boldsymbol{y},\boldsymbol{x}))(u(\boldsymbol{x})-u(\boldsymbol{y}))=0 \quad \boldsymbol{x}\in P\setminus S$$

$$u(\boldsymbol{x})=g(\boldsymbol{x}), \quad \boldsymbol{x}\in S$$

It leads to solutions which are not continuous a the known data points

Harmonic Extension on a Manifold

- As an alternative (Shi et. al. 2016), consider the continuous problem over a manifold $\mathcal{M}$ with the squared intrinsic gradient as regulariser (measure of the dimension):
  - Let u be function defined on $\mathcal{M}$ known in some regions $\Omega_1, ..., \Omega_k \subset \mathcal{M}$.
  - interpolate by solving

$$\frac{1}{2} \min_{u \in H^1(\mathcal{M})} \int_{\mathcal{M}} ||\nabla_{\mathcal{M}} u(\boldsymbol{x})||^2 \, d\boldsymbol{x},$$

    with the constraint $u(\boldsymbol{x}) = g(\boldsymbol{x})$ for all $x \in \Omega_1 \cup ... \cup \Omega_k \subset \mathcal{M}$

  - leads to the mixed Dirichlet/Neumann boundary value problem

$$\begin{aligned} -\Delta_{\mathcal{M}} u &= 0 \quad \text{on} \quad \mathcal{M} \\ u(\boldsymbol{x}) &= g(\boldsymbol{x}) \quad \text{on} \quad \partial\mathcal{M}_D \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on} \quad \partial\mathcal{M} \end{aligned}$$

    with $\partial\mathcal{M}_D$ boundary of the known data set and η the normal of $\mathcal{M}$ pointing outwards.

Point Integral Method

- Key Approximation for Laplace-Beltrami:

$$-\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\boldsymbol{y}) \bar{R}_t(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} = \frac{1}{t} \int_{\mathcal{M}} (u(\boldsymbol{x}) - u(\boldsymbol{y})) R_t(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} - 2 \int_{\partial \mathcal{M}} \frac{\partial u(\boldsymbol{y})}{\partial \eta} (g(\boldsymbol{y}) - u(\boldsymbol{y})) \bar{R}_t(\boldsymbol{x}, \boldsymbol{y})\, ds(\boldsymbol{y})$$

  with $R_t(\boldsymbol{x}, \boldsymbol{y}) = C_t R \left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t} \right), \bar{R}_t(\boldsymbol{x}, \boldsymbol{y}) = C_t \bar{R} \left(\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{4t} \right)$ with R an integrable function and $\bar{R}(r) = \int_r^\infty R(s) ds$. C_t is a normalising constant of R_t

- It follows that the boundary value problem of the previous slide can be approximated by

$$\frac{1}{t} \int_{\mathcal{M}} (u(\boldsymbol{x}) - u(\boldsymbol{y})) R_t(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} - 2 \int_{\partial \mathcal{M}_D} \frac{\partial u(\boldsymbol{y})}{\partial \eta} \bar{R}_t(\boldsymbol{x}, \boldsymbol{y})\, ds(\boldsymbol{y}) = 0$$

- However, we do not know $\frac{\partial u(\boldsymbol{y})}{\partial \eta}$ at $\partial \mathcal{M}_D$.

Point Integral Method

- We do not know $\frac{\partial u(\boldsymbol{y})}{\partial \eta}$ at $\partial \mathcal{M}_D$.
- Modifying the boundary value problem to be of Robin/Neumann

$$
\begin{aligned}
-\Delta_{\mathcal{M}} u &= 0 \quad \text{on} \quad \mathcal{M} \\
u(\boldsymbol{x}) + \beta \frac{\partial u}{\partial \eta} &= g(\boldsymbol{x}) \quad \text{on} \quad \partial \mathcal{M}_D \\
\frac{\partial u}{\partial \eta} &= 0 \quad \text{on} \quad \partial \mathcal{M}
\end{aligned}
$$

and letting $\frac{\partial u(\boldsymbol{x})}{\partial \eta} = \frac{1}{\beta}(g(\boldsymbol{x}) - u((\boldsymbol{x}))$ on $\partial \mathcal{M}_D$, leads to

$$
\frac{1}{t} \int_{\mathcal{M}} (u(\boldsymbol{x}) - u(\boldsymbol{y})) R_t(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y} - \frac{2}{\beta} \int_{\partial \mathcal{M}_D} (g(\boldsymbol{y}) - u(\boldsymbol{y})) \bar{R}_t(\boldsymbol{x}, \boldsymbol{y}) \, ds(\boldsymbol{y})
$$

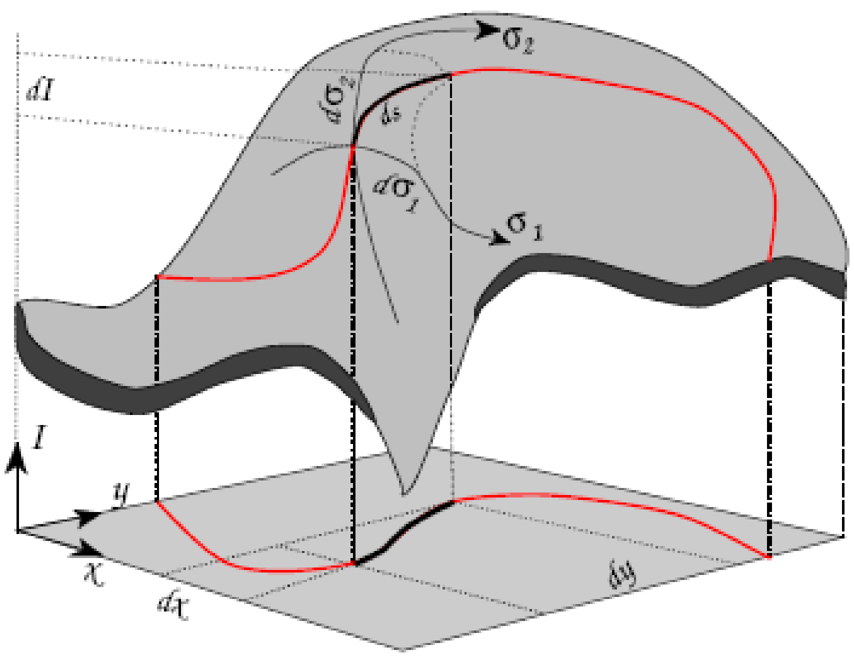
- Discretisation is straightforward

2D Images as Surfaces in 3D

- Consider image U over connected domain $D \subset \mathbb{R}^2$
- With a positive parameter β, construct the surface (image manifold)

$$\sigma(x, y) = (x, y, \beta U(x, y)) \subset \mathbb{R}^3, \quad (x, y) \in D$$

- Idea: Use surface evolution of σ to process image data U



Representation of a grey-value image by a surface (N. Sochen, R. Deriche, L. Lopez Perez 2003)

The Beltrami Flow

- consider the intrinsec diffusion (mean curvature motion)

$$\sigma_t = \Delta_\sigma \sigma$$

- keeping first two components fixed, i.e. project flow to variations of image U alone leads to

$$U_t = \frac{U_{xx}(1+\beta^2 U_y^2) + U_{yy}(1+\beta^2 U_x^2) - 2\beta^2 U_{xy} U_x U_y}{(1+\beta^2 U_y^2 + \beta^2 U_y^2)^2}$$

  the beltrami flow

- For the image U, the Beltrami flow is an edge-preserving anisotropic diffusion flow

Beltrami Flow as Gradient Descent

- Consider area of surface $\sigma(D)$ as energy

$$E[u] = \int_D \sqrt{\det \mathbf{I}_{(x,y)}} dxdy = \int_D \sqrt{1 + \beta^2 U_x^2 + \beta^2 U_y^2} dxdy$$

- Compute gradient descent for E w.r.t. following image-dependent inner product for functions V, W on D :

$$< X, Y >= \int_D \sqrt{1 + \beta^2 U_x^2 + \beta^2 U_y^2} V(x,y) W(x,y) dxdy$$

  This is the standard inner product of V and W regarded as functions on the surface σ.

- Resulting gradient descent equals Beltrami flow for grey-value image U

References

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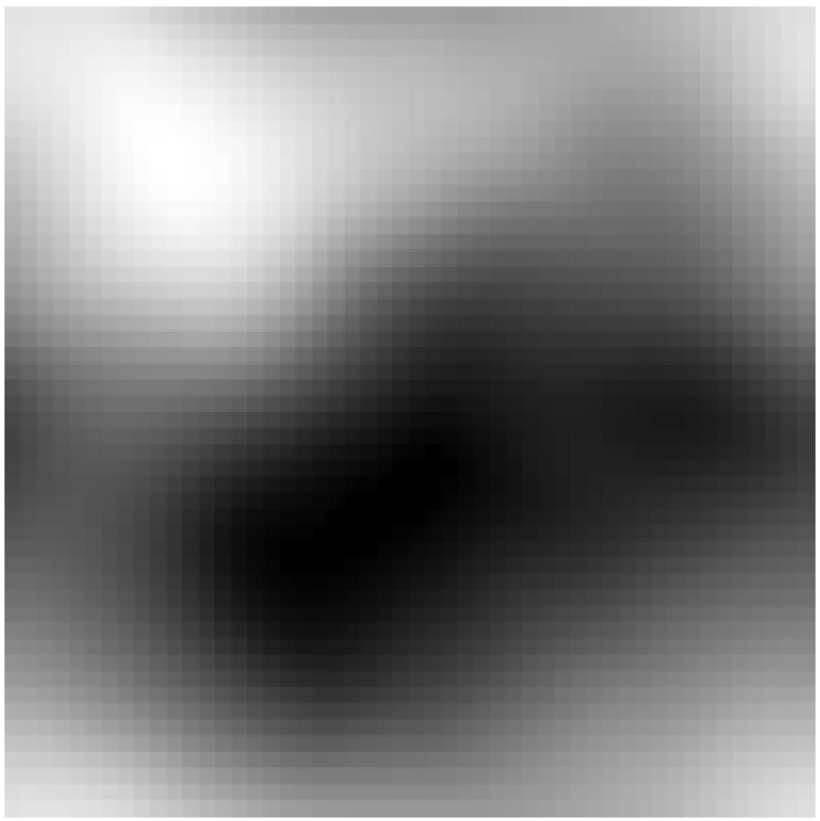
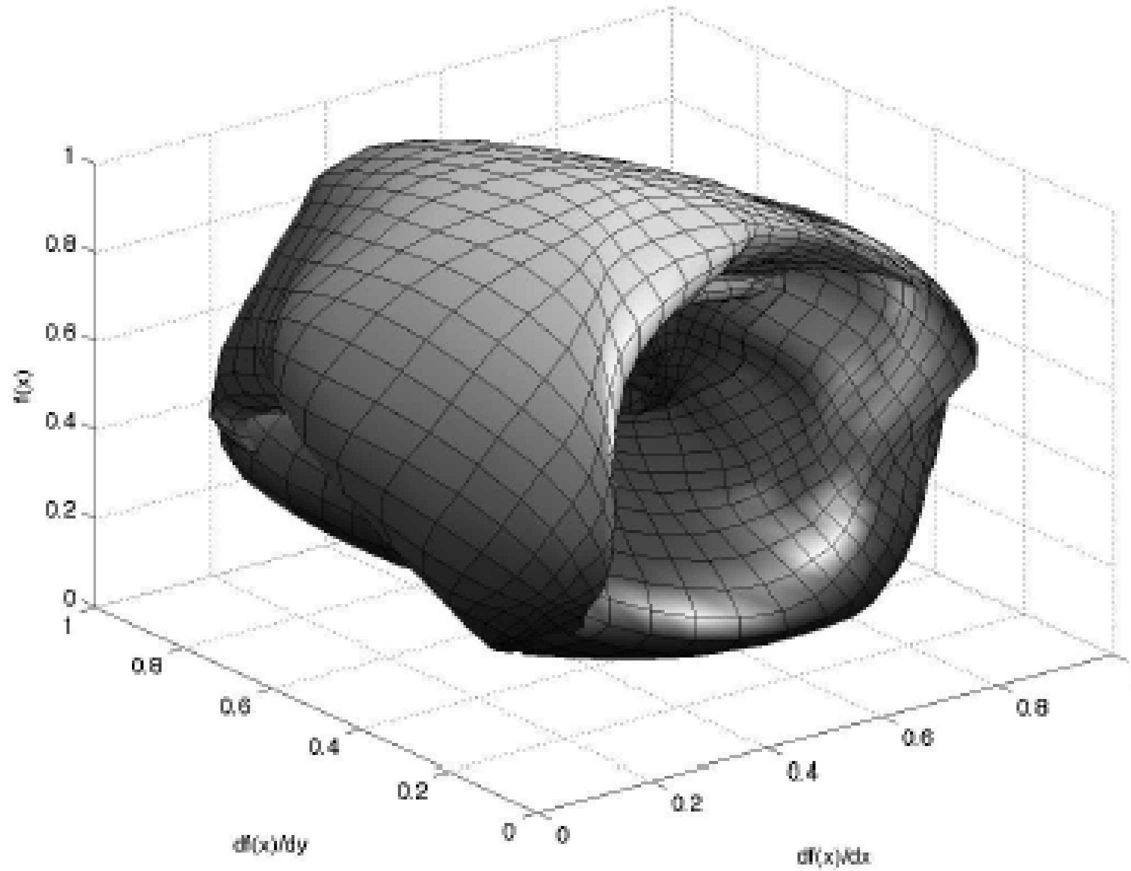
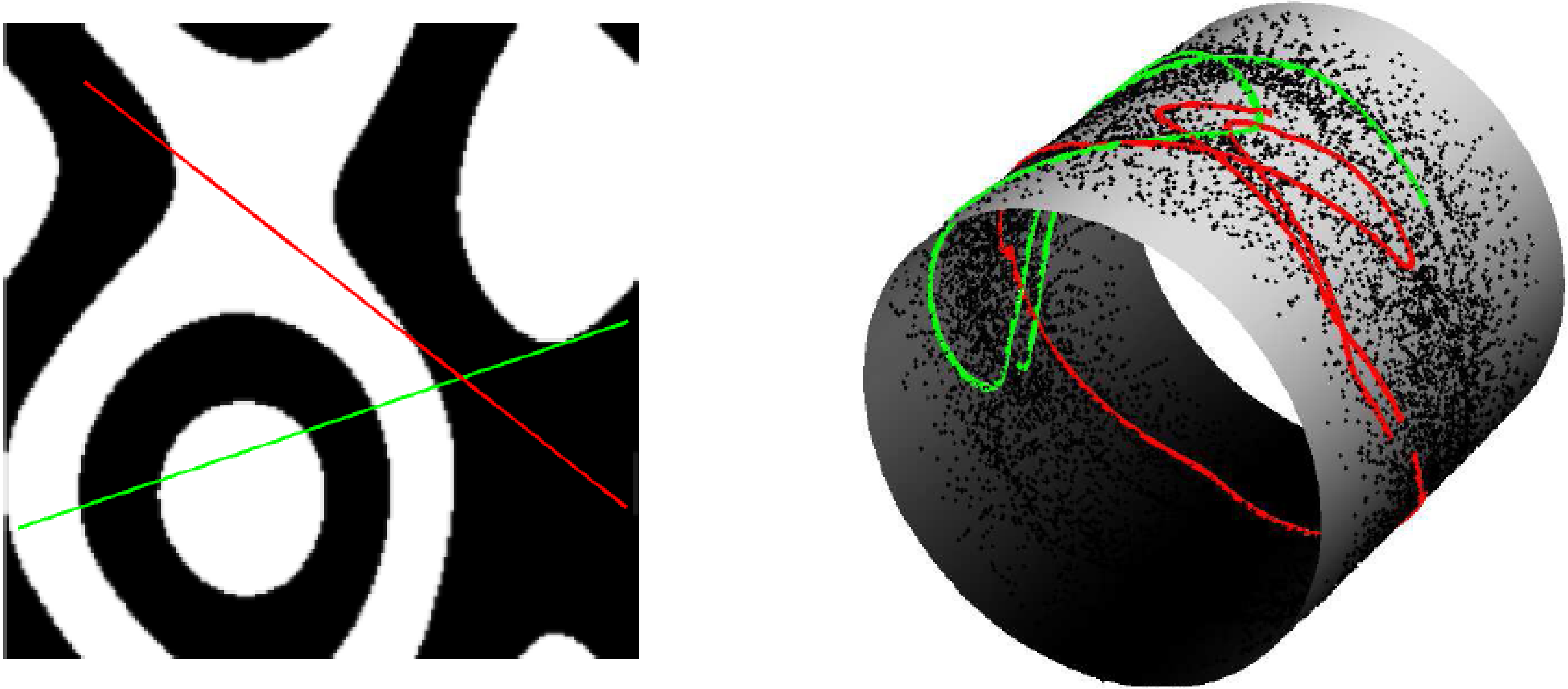
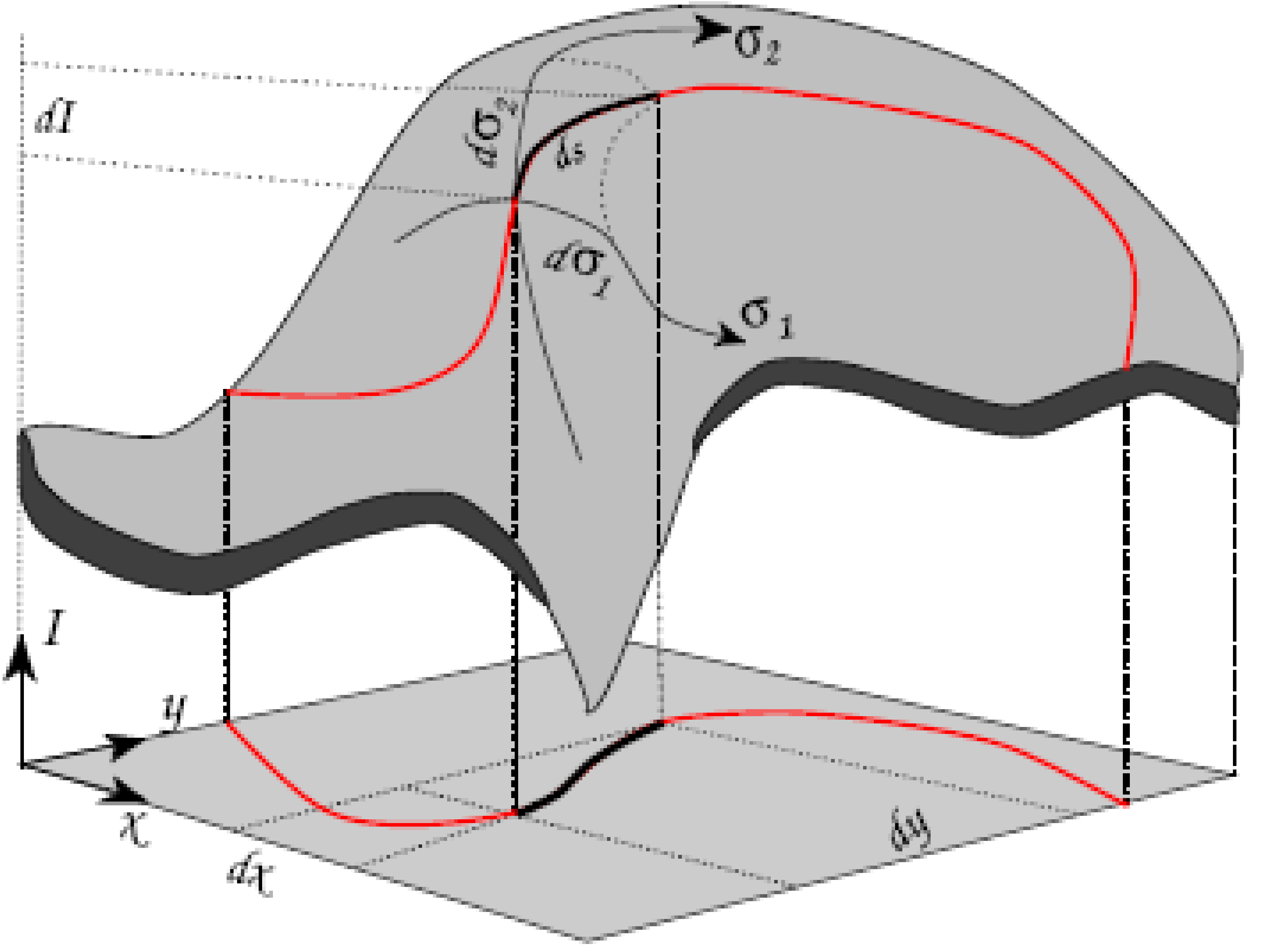


Image f



Surface $\tilde{c}_f$





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