Lecture 19

- Surface Inpainting
- Polynomial Finite Elements
- Patch Manifolds
- Low Dimensional Manifold Model

Image Inpainting

- Consider an image with missing data (greyvalue/color values known only at a subset of pixels)
- Problem: reconstruct the whole image using only the known data
- Example: use homogeneous diffusion to inpaint the missing pixels by means of solving the boundary value problem

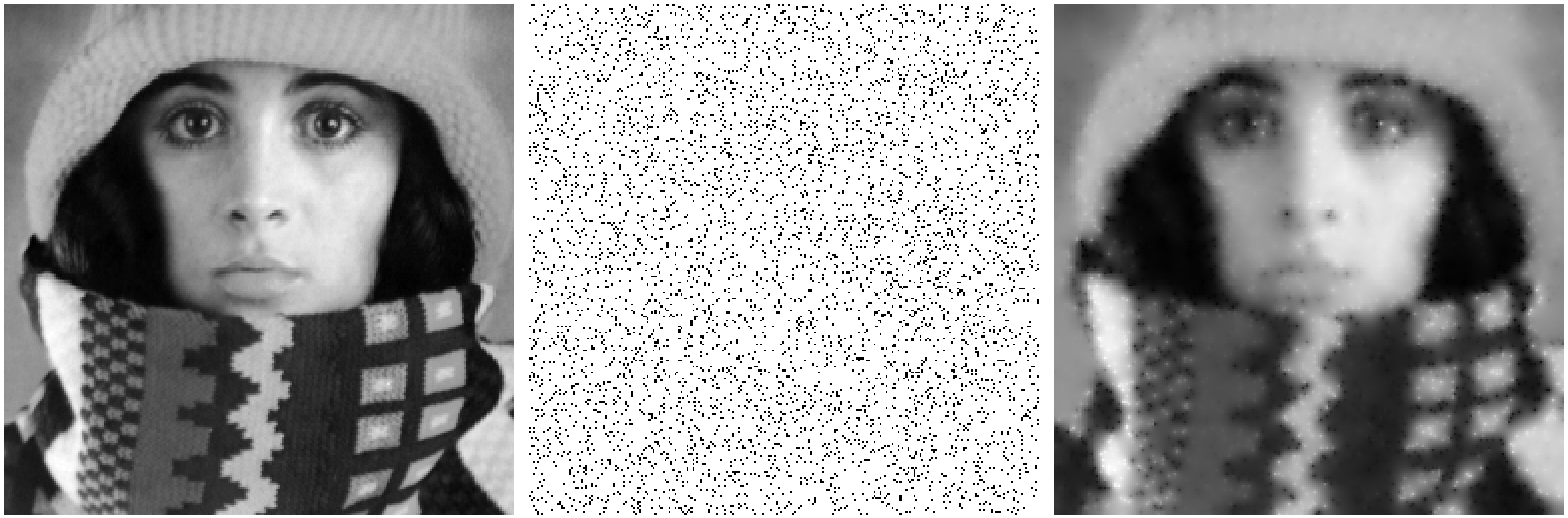
$$\begin{aligned} \Delta u &= 0 \quad \text{on} \quad \Omega \setminus K \\ u &= f \quad \text{on} \quad K \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

  with K the set of known data
- Other choices for the inpainting the mising data: biharmonic inpainting, edge enhancing diffusion, exemplar-based inpainting

Inpainting Based Compression

- basic procedure:
  - select a sparse subset of the image pixels
  - encode the image by storing only the selected pixels
  - decode the image using inpainting
- given an inpainting strategy (e.g. homogeneous linear inpainting) main difficulties:
  - select a subset of pixel which can be efficiently encoded
  - the inpainted image should be close to the original image

Inpainting Based Compression



Left to right: Original image – Set of known data – Result of homogeneous diffusion inpainting

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Inpainting Based Surface Compression

- store only a subset of the given surface
- recover the missing data with an inpainting procedure
- in case the surface is given by a triangulation we store a subset of the vertices
- in (Bae et. al. 2010) the inpainting is done solving the linear geometric diffusion equation

$$\begin{aligned} \sigma_t &= \Delta_{S(t)} u \quad \text{on} \quad S(t) \times]0, \infty[\\ \sigma &= \sigma_k \quad \text{on} \quad \partial S(t) \times]0, \infty[\\ S(0) &= S_0 \end{aligned}$$

where S_0 is some initial guess and σ_k is the known data

Surface Compression

- mean curvature operator leads to point singularities. *Solution (Bae et. al. 2010):* inpaint first the unknown data , then also at the positions of the known data
- other possible solutions: use higher order operators for the inpainting, use anisotropic diffusion
- in case of higher order operators, higher degree polynomial are better suited than piece-wise affine functions as basis for finite elements

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Surface Compression

Left to right: Input surface – inpainted surface with linear geometric diffusion – inpainted surface with modification (Bae et. al. 2010)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

The Finite Element

- $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ is called a finite element if
  - $\mathcal{K} \subset \mathbb{R}^n$ is a simply connected bounded open set with piece-wise smooth boundary the (element domain)
  - $\mathcal{P}$ is a finite-dimensional space of functions defined on $\mathcal{K}$ the (space of shape functions)
  - $\mathcal{N} = \{N_1, ..., N_k\}$ the (nodal variables) is a basis for $\mathcal{P}'$ (the dual of $\mathcal{P}$)
- A basis of $\mathcal{P}$, $\{P_1, ..., P_k\}$, is a nodal basis if it is dual to $\mathcal{N}$, namely

$$N_i(P_j) = \delta_{ij}.$$

Here $\delta_{ij} = 1$ if $i = j$ and 0 otherwise

The Finite Element

- **Lemma:** Let P be a k-dimensional linear space of functions on $\mathbb{R}^n$. Then $N_1, N_2, ..., N_k$ is a basis for $\mathcal{P}$ if and only if the following holds:

  Given that $v \in \mathcal{P}$ and $N_i(v) = 0$ for $i = 1, ..., k$, then $v = 0$

- **Lemma:** Suppose that P is a polynomial of degree $d \geq 1$ that vanishes on $\{x : L(x) = 0\}$ where L is a non-degenerate linear function. Then we can write P in the factorised form $P = LQ$ where Q is a polynomial of degree $(d - 1)$.

The Interpolant

- Let $(\mathcal{K}, \mathcal{P}, \mathcal{N})$ be a finite element with nodal basis $\{\psi_i : 1 = 1, ...k\}$. The local interpolant is given by

$$\mathcal{I}_K(v) = \sum_{i=1}^{k} N_i(v)\psi_i$$

- A subdivision of the computational domain is a finite collection of open sets K_i s.t.
  - $K_i \cap K_j =$ if $i \neq j$
  - $\cup_i \bar{K}_i = \bar{\Omega}$

- Assume Ω has a subdivision $\mathcal{T}$ of finite elements. The global interpolant is given by

$$\mathcal{I}_h(v)|_{K_i} = \mathcal{I}_{K_i}(v) \quad \forall K_i \in \mathcal{T}$$

  with h e.g. the largest diameter of all finite elements.

Continuity

- In the absence of further conditions on the subdivision it is not possible to assert the continuity of the global interpolant
- **Lemma:** Given a triangulation $\mathcal{T}$ of Ω it is possible to choose edge nodes for the corresponding elements, such that the global interpolant $I_h(v)$ belongs to $C^0(\bar{\Omega})$ for all $v \in C^m(\Omega)$, where $m = 0$ for *Lagrange elements* (Nodal points given by function values) and $m = 1$ for *Hermite elements* (Nodal points given by function values and derivatives).

Product Manifolds

- Let M and N be two manifolds of dimensions m and n
- The product manifold $M \times N := \{(x, y)\ x \in M, y \in N\}$ is a manifold of dimension $m + n$.
- charts can be constructed by taking products maps of charts of M, N

Patch Based Methods

- Non local means (Buades et. al 2005)

$$NL[u] = \frac{1}{C(x)} \int_\Omega \exp^{\frac{-(G_a * |u(x-\cdot)-u(y-\cdot)|^2)(0)}{h^2}} u(y)\, dy,$$

  with

$$C(x) = \int_\Omega \exp^{\frac{-(G_a * |u(x-\cdot)-u(y-\cdot)|^2)(0)}{h^2}} dy,$$

  where G_a is a Gaussian

- many other nonlocal methods for denoising or exemplar based inpainting (Gilboa et. al. 2009, Criminisi et. al. 2003, Arias et. al. 2011)

Patches and Manifolds

- create a patch ensemble from a given image $f : \Omega \to \mathbb{R}$: for each point $x \in \Omega$ consider a patch $L^2([-\frac{\delta}{2}, -\frac{\delta}{2}]^2)$
- main assumption: for natural images the set of patches (Lee et. al. 2003, Carlsson et. al. 2008) are well approximated by low dimensional manifolds
- examples of explicit patch manifolds (G. Peyré 2009):
  - manifold of smooth variations: C^1 images, patches well approximated by affine functions
  - manifold of cartoon images
  - manifold of locally parallel textures

Manifold of Smooth Variations

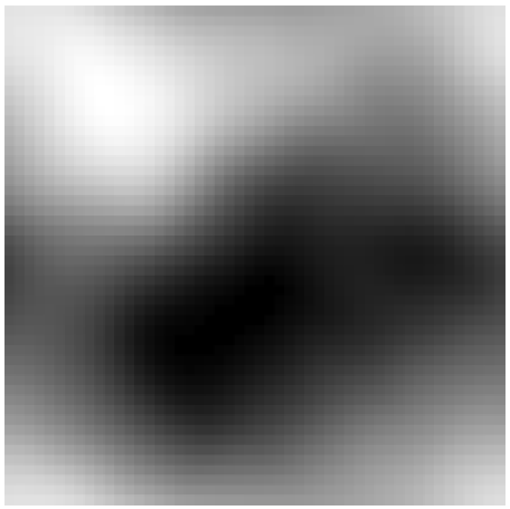
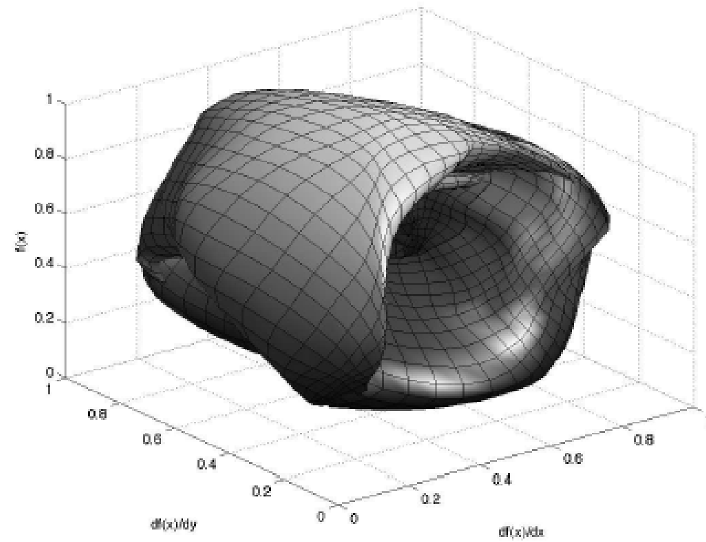


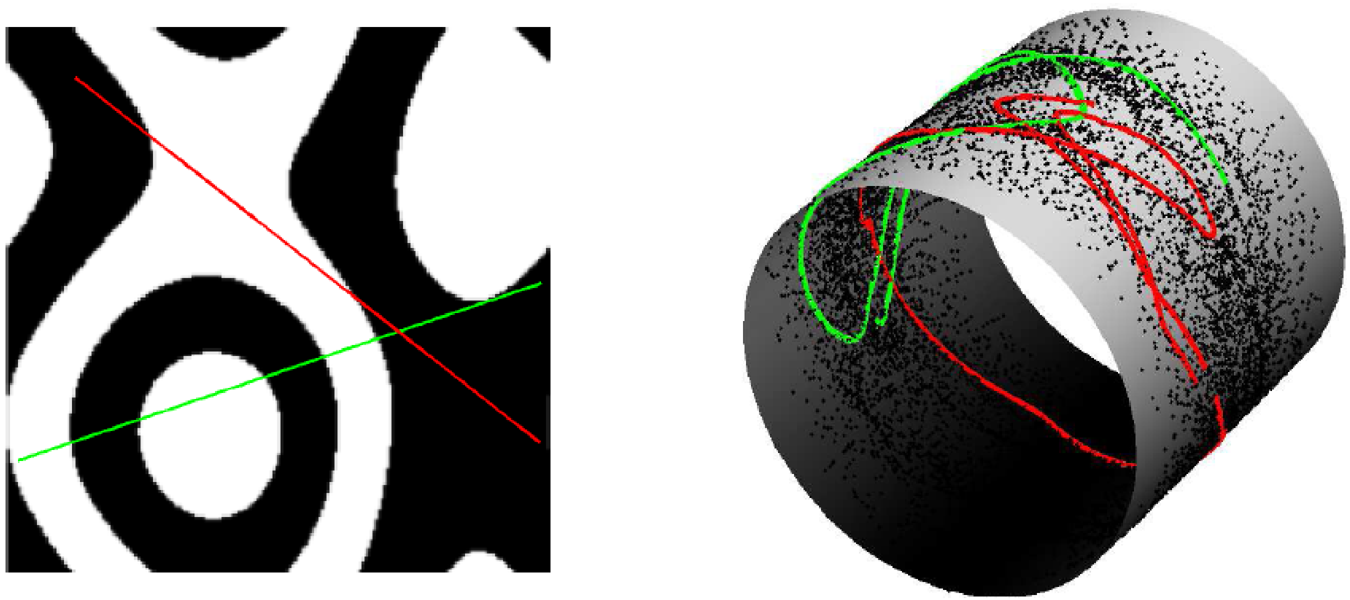
Image f



Surface $\tilde{c}_f$

Manifold of smooth images (G. Peyré 2009)

Manifold of Cartoon Images



Left to right: A cartoon image – A 3D representation of the edge manifold M (depicted in 3D as a cylinder). The two curves on the manifold corresponds to patches extracted along the two lines in the image (G. Peyré 2009)

locally parallel textures

Typical locally parallel texture (G. Peyré 2009)

Inverse Problems

- Many image processing problems can be formalized as the recovery of an image f from a set of noisy measurements Φf

$$y = \Phi f + \epsilon$$

- Φ typically accounts for some damage to the image, for instance, blurring, missing pixels, or downsampling
- In order to solve this ill-posed problem, one needs to have some prior knowledge of the image
- With the help of regularizations, many image processing problems are formulated as optimization problems, e.g.:

$$\operatorname*{argmin}_f R(f) + ||y - \Phi f||_{L^2}^2$$

Low Dimensional Manifold Regulariser

- assume that the patches of the image are well represented by a low dimensional manifold
- use the low dimensional property as a regulariser for the inverse problem:

$$\operatorname*{argmin}_{\mathcal{M}.f} \int_{\mathcal{M}} \dim(\mathcal{M}(f))(x)\, dx + \lambda ||y - \Phi f||_{L^2}^2$$

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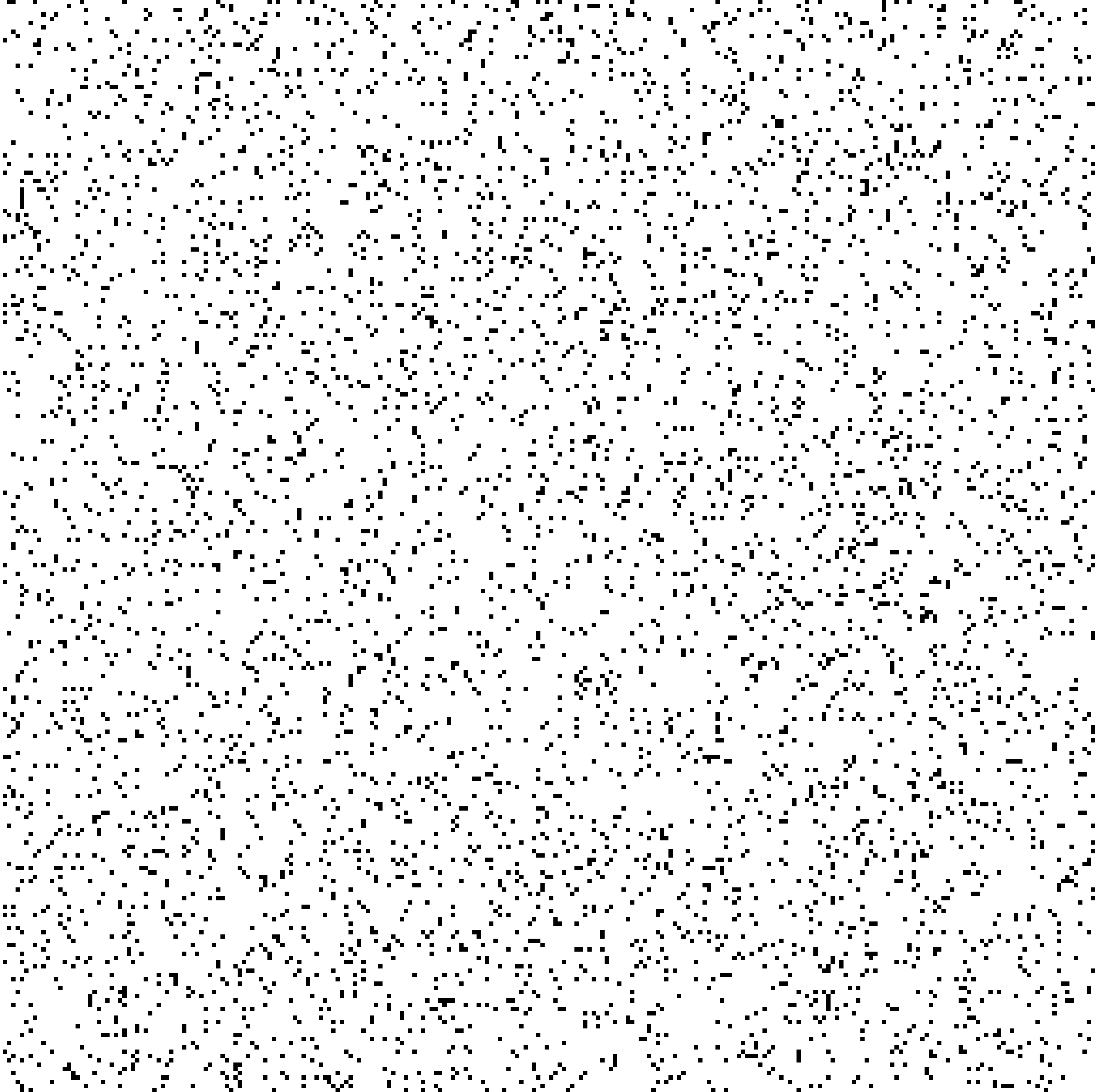


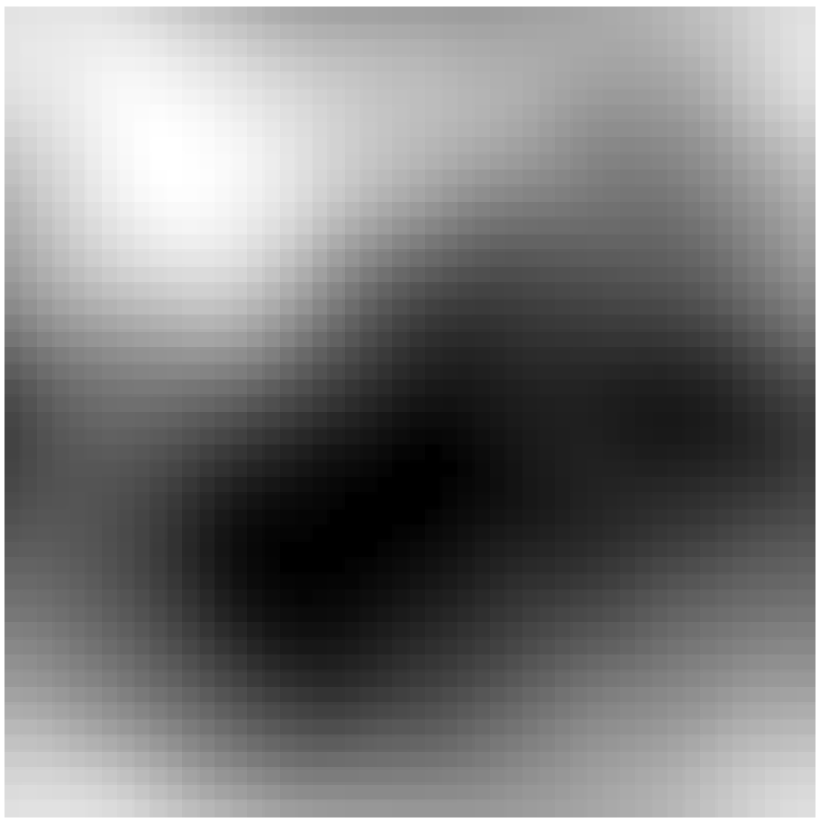
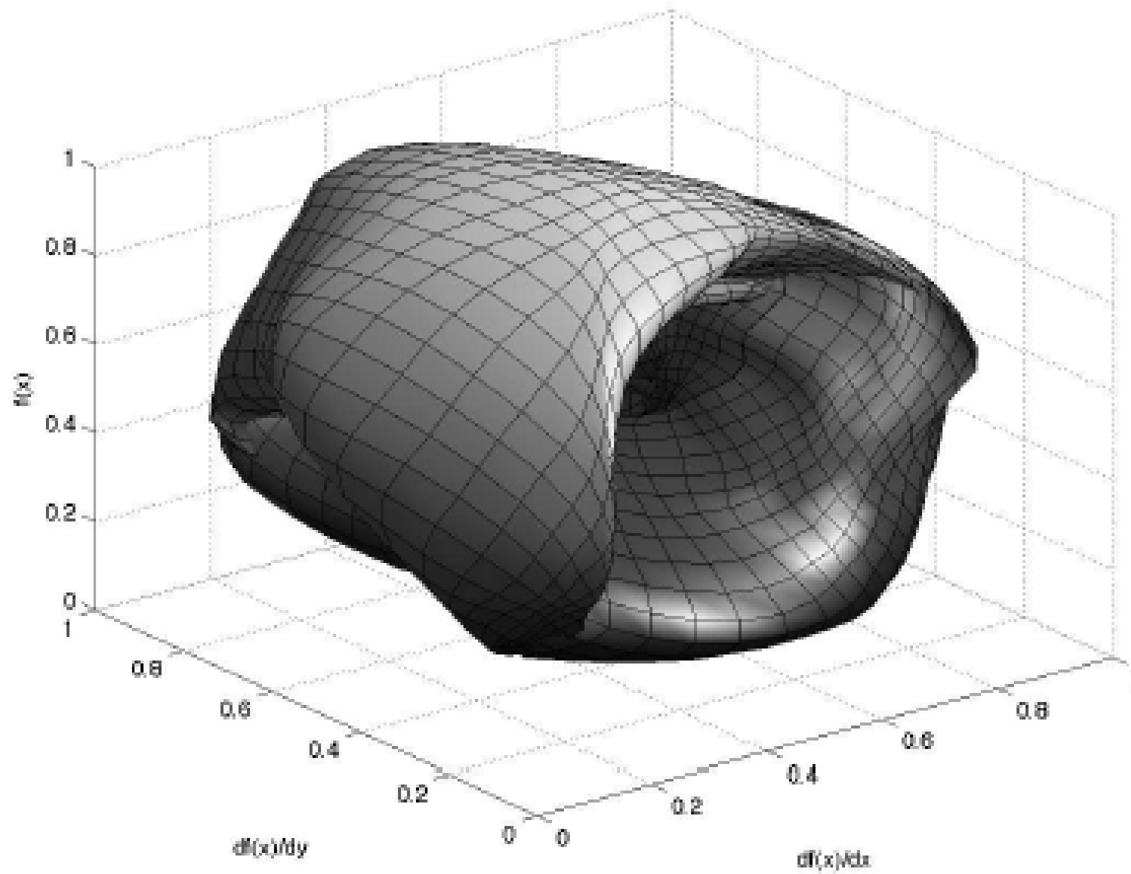


Image f



Surface $\tilde{c}_f$