Lecture 18

- Self-adjoint Problem
- Error Analysis
- Parabolic Equations
- Diffusion on Surfaces

Variational Formulation for the Self-adjoint Case

- In the special case when the boundary value problem is self-adjoint, i.e.

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad b_i(x) = 0$$

  $\forall x \in \bar{\Omega}$ the biliner functional $a(\cdot, \cdot)$ becomes symmetric.

- In this case we define the quadratic functional $J : H_0^1(\Omega) :\rightarrow \mathbb{R}$ given by

$$J(v) = \frac{1}{2}a(v, v) - l(v).$$

- **Proposition:** If $a(\cdot, \cdot)$ is symmetric bilinear, the (unique) weak solution is the unique minimiser of J over $H_0^1(\Omega)$.
- **Proposition:** Conversely, let u minimise J over $H_0^1(\Omega)$ then u is the (unique) solution of the weak boundary value problem.

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Finite dimentional case:

- Finding a weak solution u_h, of

$$\text{find} \quad u_h \in V_h \quad \text{s.t.} \quad a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$$

  corresponds to the minimisation of J over V_h, i.e

$$J(u_h) = \min_{u \in V_h} J(u)$$

Assembly of the Stiffness Matrix

Example:

- Let $\Omega \subset \mathbb{R}^2$ and consider

$$\begin{aligned} -\Delta u &= f \quad \text{on} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

- Let Ω be a bounded polygonal domain in the plane, subdivided into M triangles s.t. any pair intersect only along a complete edge, at a vertex or not at all.
- Let V_h be the continuous piecewise linear functions v_h defined on such a triangulation s.t. $v_h = 0$ on $\partial\Omega$
- u_h is characterised as the unique minimiser over V_h of the functional

$$J(v_h) = \frac{1}{2}\int_\Omega |\nabla v_h(x,y)|^2 + \int_\Omega f(x,y)v_h(x,y)$$

Assembly of the Stiffness Matrix

- Writing

$$v_h(x,y) = \sum_{i=1}^{N} V_i \phi_i(x,y)$$

with V_i the value at (x_i, y_i), ϕ_i the continuous piecewise linear basis function associated with this node and N the number of internal nodes of Ω, the problem becomes

$$\text{find} \quad \operatorname*{argmin}_{V \in \mathbb{R}^N} \frac{1}{2} V^\top A V - V^\top F,$$

with global stiffness matrix

$$A_{ij} = \int_\Omega \nabla\phi_i(x,y) \nabla\phi_j(x,y)$$

with global load vector

$$F_i = \int_\Omega f(x,y)\phi_i(x,y)$$

Assembly of the Stiffness Matrix

- The individual contribution to the functional of single triangles

$$\int_\Omega |\nabla v_h(x,y)|^2 = \sum_K \int_K |\nabla v_h(x,y)|^2$$

can be computes as

$$\int_K |\nabla v_h(x,y)|^2 = [V_1^k, V_2^k, V_3^k] \cdot A_K \cdot [V_1^k, V_2^k, V_3^k]^\top$$

- each A_K is an element stiffness matrix. Here V_i^k represent the the value of v_h at the node of the triangle K with position vector $r_i = (x_i, y_i)$ $i = 1, 2, 3$.
- We have that

$$A_k = \frac{1}{4A_{123}} \begin{pmatrix} |r_2 - r_3|^2 & (r_2 - r_3)(r3 - r1) & (r_2 - r_3)(r_1 - r_2) \\ . & |r_1 - r_3|^2 & (r_3 - r_1)(r_1 - r_2) \\ \text{symm} & . & |r_1 - r_2|^2 \end{pmatrix}$$

with A_{123} the area of the triangle.

Assembly of the Stiffness Matrix

- Let $1, 2..., N$ index the inner nodes and $N+1, N+2, ..., N^*$ index the boundary nodes. Then

$$u_h(x,y) = \sum_{i=1}^{N^*} V_i \phi_i(x,y)$$

with $V_j = 0$ for $j = N+1, ..., N^*$.

- The full stiffness matrix can be assembled by

$$A^* = \sum_{k=1}^{M} L^k A^k (L^k)^\top,$$

where L^k are appropriate $N^* \times 3$ boolean matrices

- Once A^* is known, we erase the last $N^* - N$ rows and columns of A to obtain the global stiffness matrix A (analogous procedure for F)

Error Analysis

- consider again the general elliptic boundary problem of the previous lecture. How can we measure if $u_h \in V_h$ is a good approximation of the weak solution $u \in H_0^1(\Omega)$?
- The Galerkin orthogonality

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

leads to

- **Lemma** (Céa's Lemma): The finite element approximation u_h to $u \in H_0^1(\Omega)$, is the near-best fit to u in the norm $|| \cdot ||_{H_0^1(\Omega)}$, i.e.

$$||u - u_h||_{H_0^1(\Omega)} \leq \frac{c_1}{c_0} \min_{v_h \in V_h} ||u - v_h||_{H_0^1(\Omega)}$$

Other Boundary Conditions

Example: Mixed Boundary Conditions

$$
\begin{aligned}
-\Delta u &= f \quad x \in \Omega \\
u &= 0 \quad x \in \Gamma_1 \\
\frac{\partial u}{\partial \nu} &= g \quad x \in \Gamma_2,
\end{aligned}
$$

With $f \in L^2(\Omega), g \in L^2(\Gamma_2)$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$

1 2 3 4 5 6 7 8 9 10 11 12 13

Parabolic Equations

- Consider the initial value problem

$$\begin{aligned} \partial u(x,t) - \Delta u(x,t) &= f(x,t) \quad x \in \Omega \subset \mathbb{R}^2 \\ u(x,t) &= 0 \quad x \in \partial\Omega,\ 0 \le t \le T \\ u(0,x) &= u_0(x) \quad x \in \Omega, \end{aligned}$$

- Being ϕ_i basis correspoding to the nodes of a triangulation and using the representation

$$u_h(x,t) = \sum_{i=1}^{N} \xi_i(t)\phi_i(x)$$

the weak problem becomes finding the coefficients $\xi_i(t)$ s.t.

$$\xi'(t) + M^{-1}S\xi(t) = b(t), \quad t > 0 \quad \xi(0) = \eta$$

with $M_{ij} = (\phi_i, \phi_j)_{L^2(\Omega)}$ the mass matrix, S the stiffness matrix and b the load vector and $\sum_{i=1}^{N} \eta_i\phi_i(x)$ is an approximation of u_0

Example: Anisotropic Surface Diffusion

- Given an initial compact embedded surface M_0 embedded in $\mathbb{R}^3$, compute a family of sufaces $\mathcal{M}(t)_{t\geq 0}$ with corresponding coordinate mappings $x(t)$ s.t.

$$\partial x - div_{\mathcal{M}(t)}(a_\epsilon \nabla_{\mathcal{M}(t)} x) = f \quad \text{on} \quad \{t > 0\} \times \mathcal{M}(t)$$
$$\mathcal{M}(0) = \mathcal{M}_0$$

  where the diffusion tensor a_ϵ is supposed to be a symmetric, positive definite, linear mapping on the tangent space $T_x\mathcal{M}$

- in weak formulation

$$\int_{\mathcal{M}(t)} \theta \partial_t x + \int_{\mathcal{M}(t)} g(a_\epsilon \nabla_{\mathcal{M}(t)} X(t), \nabla_{\mathcal{M}(t)} \theta) = \int_{\mathcal{M}(t)} \theta f(t)$$

  $\forall \theta \in C^\infty(\mathcal{M}(t))$ with g the metric of $\mathcal{M}(t)$.

Example: Anisotropic Surface Diffusion

- The space discrete problem is to find a family of discrete successively smoothed and sharpened surfaces $\mathcal{M}_h(t)$ with coordinate maps $X(t)$, the weak formulation is

$$\int_{\mathcal{M}_h(t)} \Theta \partial_t X(t) + \int_{\mathcal{M}_h(t)} g(a_\epsilon \nabla_{\mathcal{M}_h(t)} X(t), \nabla_{\mathcal{M}_h(t)} \Theta) = \int_{\mathcal{M}_h(t)} \Theta f(t)$$

- a backwards Euler scheme leads to (Clarez et. al. 2000)

$$(M^n + \tau L^n(A_\epsilon^n))\bar{X}^{n+1} = M^n \bar{X}^n + \tau M^n \bar{F}^n$$

for the new vertex positions $\bar{X}^{n+1}$ at time $t^{n+1} = \tau(n+1)$.
Here M is the mass matrix

$$M_{ij}^n = \int_{\mathcal{M}_h^n} \Phi_i \Phi_j$$

and

$$L_{ij}^n = \int_{\mathcal{M}_h^n} g(A_\epsilon^n \nabla_{\mathcal{M}_h^n} \Phi_i, \nabla_{\mathcal{M}_h^n} \Phi_j)$$

the nonlinear stiffness matrix with linear nodal basis $\{\Phi_i\}$

References

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- G. Dziuk, C. M. Elliott, Surface Finite Elements For Parabolic Equations, Journal of Computational Mathematics, 2007