Lecture 17

- Weak Solutions of PDEs
- Introduction to Finite Elements
- Variational Formulations: Self-Adjoint Case

Weak Soultions

- up to now we have assumed that the PDEs we considered have smooth coefficients/solutions
- what if we are dealing with a PDE which has initial data (boundary data), or coefficients which are not smooth, e.g.

$$
\begin{aligned}
-\Delta u &= \text{sgn}(\frac{1}{2} - |x|) && \text{in} \quad \Omega \\
u &= 0 && \text{in} \quad \partial\Omega
\end{aligned}
$$

  with $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$

- multiply by a smooth compact supported function ϕ and integrate by parts

$$
\int_\Omega \nabla u \nabla \phi = \int_\Omega \text{sgn}(\frac{1}{2} - |x|)\phi
$$

  this expression makes sense even if u is not twice differentiable

Classical solutions

- Consider the homogeneous Dirichlet boundary value problem over a bounded open domain $\Omega \subset \mathbb{R}^n$ (here $\bar{\Omega}$ denotes its closure)

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=0}^{n} b_i(x)\frac{\partial u}{\partial x_i} + uc(x) = f(x) \quad in \quad x \in \Omega$$

$$u(0,x) = 0 \quad in \quad x \in \partial\Omega.$$

with $a_{i,j} \in C^1(\bar{\Omega})$, $b_i, c, f \in C(\bar{\Omega})$ and $\exists \tilde{c} > 0$ s.t.

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \tilde{c}\sum_{i=1}^{n} |\xi_i|^2$$

for all $x \in \bar{\Omega}$, $(\xi_1, ..., \xi_n) \in \mathbb{R}^n$ (uniform ellipticity).

- A function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solving the problem is called classical solution
- What if the coefficients are not smooth? we consider weak solutions

Weak solutions

- Let $a, b, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. A function $u \in H_0^1(\Omega)$ satisfying

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=0}^{n} \int_\Omega b_i(x) \frac{\partial u}{\partial x_i} v + \int_\Omega c(x) uv = \int_\Omega f(x) v \qquad (1)$$

  $\forall v \in H_0^1(\Omega)$, is called a weak solution of the homogeneous Dirichlet boundary value problem (here all the derivatives are understood in the weak sense).

- classical solutions are also a weak solution but the converse is not true

Existence of a weak solution

- The weak formulation can be written as

$$\text{find} \quad u \in H_0^1(\Omega) \quad \text{s.t.} \quad a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega) \tag{2}$$

with the notation

$$a(w,v) = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) \frac{\partial w}{\partial x_j} \frac{\partial v}{\partial x_i} + \sum_{i=0}^{n} \int_\Omega b_i(x) \frac{\partial w}{\partial x_i} v + \int_\Omega c(x) w v$$

$$l(v) = \int_\Omega f v$$

Existence of a weak solution

- **Proposition:** Assuming that

$$c(x) - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial b_i}{\partial x_i} \geq 0, \qquad \forall x \in \bar{\Omega},$$

  there existence a unique weak solution $u \in H_0^1(\Omega)$ for the homogeneous Dirichlet boundary value problem (1), equiv. (2). This is a consequence of the general result:

- **Theorem (Lax Milgram):** Let V be a real Hilbert space with norm $||\cdot||_V$. Let $a(\cdot,\cdot)$ be a bilinear functional on $V \times V$ and $l(\cdot)$ a linear funtional s.t.

  i) $\exists c_0 > 0,\ a(v,v) \geq c_0||v||_V^2$

  ii) $\exists c_1 > 0,\ a(v,w) \leq c_1||v||_V||w||_V$

  iii) $\exists c_2 > 0,\ l(v) \leq c_2||v||_V$

  for all $v, w \in V$

Weak solutions: Example

- The problem

$$-\Delta u = \mathsf{sgn}(\frac{1}{2} - |x|) \quad x \in \Omega$$
$$u = 0 \quad x \in \partial\Omega$$

does not have a classical solution but does have a weak solution $u \in H_0^1(\Omega)$.

Basic Procedure

- First step is to convert the problem into its weak formulation:

$$\text{find} \quad u \in V \quad \text{s.t.} \quad a(u,v) = l(v) \quad \forall v \in V$$

  where V is the solution space

- Then replace V by a finite-dimensional subspace $V_h \subset V$ associated with a subdivision given by a discrete representation of the domain Ω and consider

$$\text{find} \quad u_h \in V_h \quad \text{s.t.} \quad a(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h$$

  e.g. V_h the continuous piece-wise polynomial functions of a fixed degree w.r.t. a triangulation of the domain

- If $dim(V_h) = N(h)$ and $V_h = span\{\phi_1, \phi_2, ..., \phi_{N(h)}\}$ the problem is to find $U_1, ..., U_{N(h)} \in \mathbb{R}$ s.t.

$$\sum_{i=1}^{N(h)} a(\phi_i, \phi_j) U_j = l(\phi_j) \qquad j = 1, ..., N(h)$$

Example

- Consider the boundary value problem

$$-(p(x)u')' + q(x)u = f(x), \quad x \in (0,1)$$
$$u(0) = 0,\ u(1) = 0$$

  where $p \in C[0,1], q \in C[0,1], f \in L^2(0,1)$ with $p(x) \geq \tilde{c} > 0$ and $q(x) \geq 0$ for all x in $[0,1]$.

- The weak formulation is given by: find $u \in H_0^1(0,1) := H_0^1((0,1))$ such that

$$\int_0^1 p(x)u'(x)v'(x) + \int_0^1 q(x)u(x)v(x) = \int_0^1 f(x)v(x)$$

  for all $v \in H_0^1(0,1)$

- we can approximated the solution with continuous piece-wise linear functions using a uniform subdivision if the interval $\bar{\Omega} = [0,1]$

Variational Formulation for the Self-adjoint case

- In the special case when the boundary value problem is self-adjoint, i.e.

$$a_{ij}(x) = a_{ji}(x) \quad \text{and} \quad b_i(x) = 0$$

  $\forall x \in \bar{\Omega}$ the biliner functional $a(\cdot, \cdot)$ becomes symmetric.

- In this case we define the quadratic functional $J : H_0^1(\Omega) :\rightarrow \mathbb{R}$ given by

$$J(v) = \frac{1}{2} a(v, v) - l(v).$$

- **Proposition:** If $a(\cdot, \cdot)$ is symmetric bilinear, the (unique) weak solution is the unique minimiser of J over $H_0^1(\Omega)$.
- **Proposition:** Conversely, let u minimise J over $H_0^1(\Omega)$ then u is the (unique) solution of the weak boundary value problem.

Other Boundary Conditions

Example: Mixed Boundary Conditions

$$\begin{aligned} -\Delta u &= f \quad && x \in \Omega \\ u &= 0 \quad && x \in \Gamma_1 \\ \frac{\partial u}{\partial \nu} &= g \quad && x \in \Gamma_2, \end{aligned}$$

With $f \in L^2(\Omega), g \in L^2(\Gamma_2)$ and $\partial\Omega = \Gamma_1 \cup \Gamma_2$