Lecture 16

- Surfaces Smoothing
- Diffusion of Surfaces
- Finite Elements: Weak Solutions of PDEs

Problem Statement

- Measured surface data, e.g., originating from 3D laser scanning
- Data are noisy
- Goal: Denoise these data
- Image processing ideas can in principle be used

Surface Mean Curvature Motion

- Given by the surface evolution
  $$\sigma_t = 2\mathbf{H}\overrightarrow{n}$$
  with $\mathbf{H} = (\kappa_1 + \kappa_2)/2$, the mean curvature
- Let function $U : \Omega \times [0, T] \to \mathbb{R}$, $\Omega \subset \mathbb{R}^3$ give a level set representation:
  - Mean curvature motion is equivalent to the 3D evolution
    $$U_t = 2\mathbf{H}||\nabla U||$$
  - Level set formulation can be rewritten into
    $$U_t = ||\nabla U|| div \left(\frac{\nabla U}{||\nabla U||} \right)$$

Remark: Here $\overrightarrow{n}$ is the surface normal pointing inwards, thus $\overrightarrow{n} = -\frac{\nabla U}{||\nabla U||}$.

Surfaces Mean Curvature Motion, Equivalences

- Equivalent description of the image evolution as smoothing along level sets

$$U_t = U_{\xi\xi} + U_{\eta\eta}$$

where $\xi(x, y, z), \eta(x, y, z) \perp \nabla U(x, y, z)$ are orthogonal unit vectors $\xi(x, y, z) \perp \eta(x, y, z)$

- Equivalent reformulation of surface evolution as smoothing of surface coordinates

$$\sigma_t = \sigma_{uu} + \sigma_{vv}$$

if σ_u, σ_v are unit vectors and $\sigma_u \perp \sigma_v$

- Equivalent variational description: gradient descent for surface area

$$E[\sigma] = \int_S dS = Area(S)$$

Mean Curvature Motion as Geometric Diffusion Process

- Mean curvature motion can be formulated as

$$\sigma_t(u_0, v_0, t_0) = \sigma_{uu}(u_0, v_0, t_0) + \sigma_{vv}(u_0, v_0, t_0)$$

  if the surface evolution is parametrised such that $\sigma_u(u_0, v_0, t_0)$ and $\sigma_v(u_0, v_0, t_0)$ are orthogonal unit vectors

- Using the intrinsic differential operators on σ, this can be written as

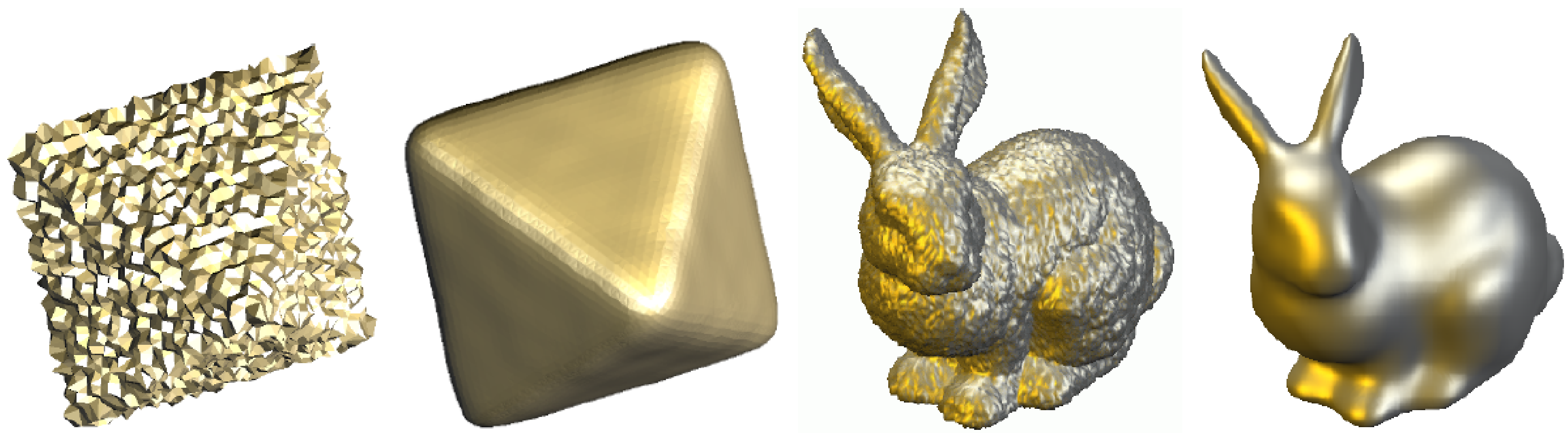
$$\sigma_t = \Delta_S \sigma$$

  i.e. linear diffusion of Note that this evolution acts channel-wise: $\partial_t \sigma_i = \Delta_S \sigma_i, i = 1, 2, 3$

- In this context, mean curvature motion is therefore also denoted as (linear) geometric diffusion

- Note that this process is linear and isotropic as intrinsic diffusion of the surface but is anisotropic within the surrounding space $\mathbb{R}^3$

Geometric Diffusion: Examples



Left to right: Noisy octahedron smoothed by mean curvature motion noisy Stanford bunny smoothed by mean curvature motion (Clarenz, Diewald, Rumpf 2000)

Isotropic Nonlinear Image Diffusion

$$\begin{aligned} u_t = div(g(|\nabla u_\rho|^2)\nabla u) \quad & in \quad \Omega \\ u(\cdot, 0) = f \quad & in \quad \Omega \\ g(|\nabla u_\rho|)\frac{\partial u}{\partial \overrightarrow{\nu}} = 0 \quad & in \quad \partial\Omega \end{aligned}$$

with u_ρ is a slightly Gaussian smoothed version of u and ν is the outer unit normal at the boundary of Ω

- $|\nabla u_\rho|^2$ works like a fuzzy edge detector
- diffusivity g is decreasing in $|\nabla u_\rho|^2$, e.g.

$$g(s) = \frac{1}{1 + \frac{s^2}{\lambda^2}}$$

- gaussian smoothing inside the diffusivity leads to a well-posed parabolic PDE

Isotropic Nonlinear Geometric Diffusion

- Analogously to diffusion of image data, a diffusivity function can be introduced into the geometric diffusion equation
- The diffusivity function should suppress diffusion depending on the geometric structure of the surface
- Parameters of the diffusivity function should be geometric invariants. The relevant invariants describing the geometric structure of the surface are the principal curvatures κ_1, κ_2
- The resulting isotropic nonlinear geometric diffusion reads in its basic form

$$\sigma_t = div_S(g(\kappa_1, \kappa_2)\nabla_S \sigma)$$

(i.e., $\partial_t \sigma_i = div_S \left(g(\kappa_1, \kappa_2)\nabla_S \sigma_i\right), i = 1, 2, 3$)

- Scalar-valued function g should be decreasing w.r.t. curvatures

Isotropic Nonlinear Geometric Diffusion

- To reduce sensitivity to noise and improve stability, one can let the diffusivity depend on a pre–smoothed surface $\tilde{\sigma} = \sigma_\rho$ which is e.g. the result of a short period of linear geometric diffusion

$$\sigma_\rho = \tilde{\sigma}(t = \frac{\rho^2}{2}), \qquad \tilde{\sigma}_t = \Delta_S \tilde{\sigma}, \qquad \tilde{\sigma}(t = 0) = \sigma$$

- Resulting evolution:

$$\sigma_t = div_S(g(\kappa_1(\tilde{\sigma}), \kappa_2(\tilde{\sigma}))\nabla_S \sigma)$$

- Possible diffusivity function (Clarenz et al. 2000) is $g = G\left(\sqrt{k_1^2 + k_2^2}\right)$ where:

$$G(s) = \begin{cases} \frac{1}{1+\frac{(|s|-\lambda\theta)^2}{(1-\theta)^2\lambda^2}} & |s| > \theta\lambda, \\ 1 & \text{otherwise} \end{cases}$$

- λ serves as a threshold value for the identification of edges and mean curvature motion is done for $\lambda\theta < 0$

Level Set Formulation of Isotropic Nonlinear Geometric Diffusion

- Consider function $U : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^3$
- Isotropic nonlinear geometric diffusion of the level sets of U reads

$$U_t = ||\nabla U||\, div \left(g(\kappa_1(\tilde{\sigma}), \kappa_2(\tilde{\sigma})) \frac{\nabla U}{||\nabla U||} \right)$$

- Corresponding natural boundary condition:

$$g(\kappa_1(\tilde{\sigma}), \kappa_2(\tilde{\sigma})) \frac{\partial U}{\partial \overrightarrow{\nu}} = 0,$$

  with $\overrightarrow{\nu}$ outer normal over $\partial\Omega$
- Applications:
  - Implementation of surface smoothing without parametrisation of surfaces
  - Smoothing of volume data
- Simplified pre-smoothing: Specifically in the level-set framework, the geometrically "correct" geometric pre-smoothing is often replaced by the simpler linear 3D diffusion. Since the pre-smoothing parameter small, the difference in results is small

Anisotropic Nonlinear Images Diffusion

- anisotropic diffusion takes into account the direction of the local structure of the image
- this cannot be achieved with a scalar-valued diffusivity g
- g is replaced by a positive definite symmetric 2×2 matrix, the diffusion tensor D:

$$\partial u = div(D \nabla u)$$

- the local image structure specifies the eigenvectors and eigenvalues of D

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

Anisotropic Nonlinear Images Diffusion

Example: Edge Enhancing Diffusion

$$\begin{aligned} u_t = div(D(\nabla u_\rho)\nabla u) \quad & in \quad \Omega \\ u(\cdot, 0) = f \quad & in \quad \Omega \\ \langle D(\nabla u_\rho)\nabla u, \nu\rangle = 0 \quad & in \quad \partial\Omega \end{aligned}$$

with u_ρ is a slightly Gaussian smoothed version of u and ν the unit normal at the boundary of Ω

- the diffusion tensor D is chosen s.t.
  - its normalised eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ satisfy $\boldsymbol{v}_1 || \nabla u_\rho$ and $\boldsymbol{v}_2 \perp \nabla u_\rho$
  - the corresponding eigenvalues are $\lambda_1 = g(|\nabla u_\rho|^2)$ and $\lambda_2 = 1$
- the eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ and their eigenvalues λ_1, λ_2 determine the diffusion tensor:

$$D(\nabla u_\rho) = \lambda_1 \boldsymbol{v}_1 \boldsymbol{v}_1^\top + \lambda_2 \boldsymbol{v}_2 \boldsymbol{v}_2^\top$$

Anisotropic Nonlinear Geometric Diffusion

- Introducing anisotropic diffusion tensor allows different diffusivities parallel and perpendicular to edges
- Diffusion tensor depends on curvatures and curvature directions of pre-smoothed surface
- For example (Clarenz et al. 2000):

$$D = a_\rho := C \begin{pmatrix} G(k_{\rho,1}) & 0 \\ 0 & G(k_{\rho,2}) \end{pmatrix} C^\top$$

  with G like in slide 9, $k_{\rho,1}, k_{\rho,2}$ are the principal curvatures of the smoothed surface $\sigma_\rho = \tilde{\sigma}$ and C is matrix of its principal curvature

- Resulting anisotropic nonlinear geometric diffusion

$$\tilde{\sigma}_t = div(D(\tilde{\sigma})\nabla_S \sigma)$$

Anisotropic Nonlinear Geometric Diffusion

- Leads to the definition of a generalised mean curvature

$$\mathbf{H}_{a_\rho} := \mathrm{tr}(a_\rho)$$

  the $a_\rho -$ mean curvature

- if σ is a solution of the anisotropic equation, then

$$\frac{d}{dt} Area(\sigma(t)) = -\int_{\sigma(t)} \mathbf{H}\mathbf{H}_{a_\rho}$$

$$\frac{d}{dt} Vol(\sigma(t)) = -\int_{\sigma(t)} \mathbf{H}_{a_\rho}$$

  which reflects one smoothing aspect of the model

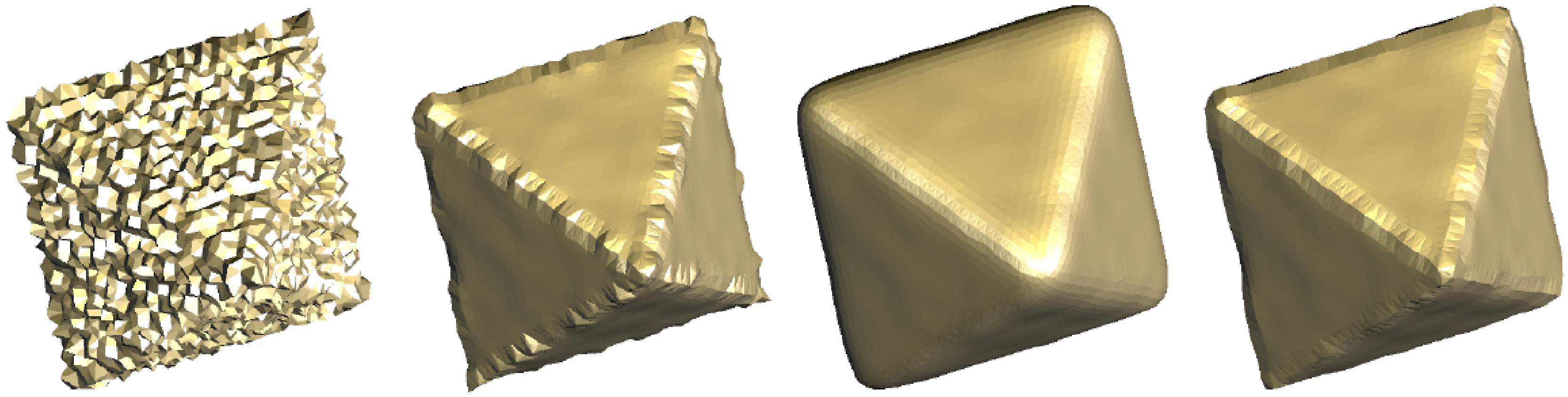
Level Set Formulation of Nonlinear Geometric Diffusion

- Level set form:

$$U_t = ||\nabla U||\, div \left(D(\tilde{U}) \frac{\nabla U}{||\nabla U||} \right)$$

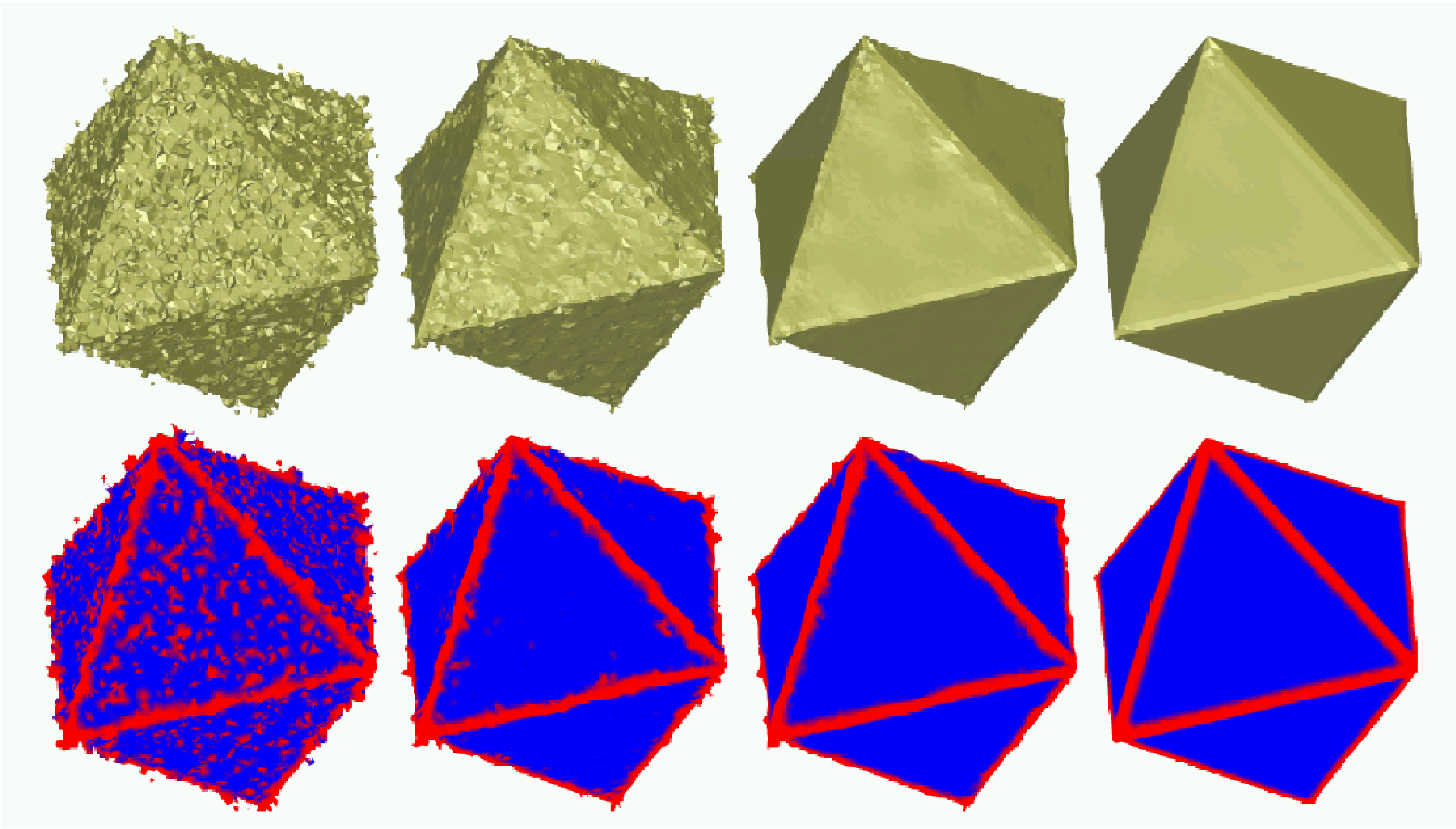
where $\tilde{U}$ is pre–smoothed function

Geometric Diffusion Examples



Left to right: A noisy octahedron surface, smoothed by isotropic nonlinear 3D diffusion (Perona-Malik model), mean-curvature motion, and anisotropic geometric diffusion (U. Clarenz, U. Diewald, M. Rumpf 2000)

Denoising by Anisotropic Nonlinear Geometric Diffusion Examples

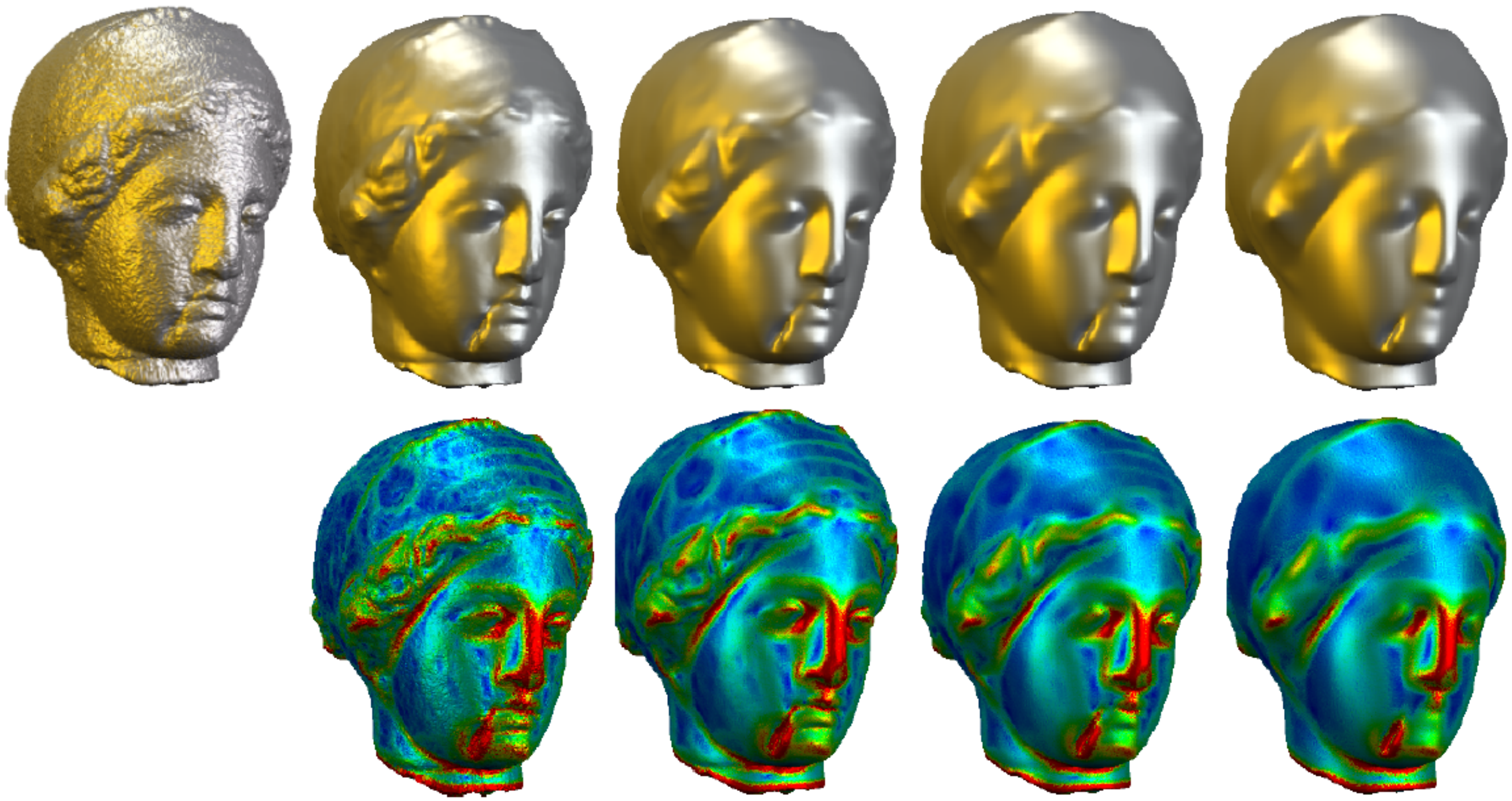


Top: Evolution of a noisy octahedron under anisotropic geometric diffusion.
Bottom: Same with colour-coded principal curvature (T. Preusser, M. Rumpf 2002)

Geometric Diffusion Examples

Left to right: Noisy Stanford bunny, smoothed by mean-curvature motion, by anisotropic geometric diffusion, with colour-coded principal curvature (U. Clarenz, U. Diewald, M. Rumpf 2000)

Denoising by Anisotropic Nonlinear Geometric Diffusion Examples



Top left: Noisy 3D scan image of a sculpture. **Top row, to right:** Filtered by anisotropic geometric diffusion, threshold $\lambda = 10$, pre–smoothing $\rho = 0.02$, at evolution times $T = 0.0002, T = 0.0004, T = 0.0006, T = 0.0008$. **Bottom:** Same filtered surfaces as above, with colour-coded principal curvatures (U. Clarenz, U. Diewald, M. Rumpf 2000)

Weak Soultions

- up to now we have assumed that the PDEs we considered have smooth coefficients/solutions
- what if we are dealing with a PDE which has initial data (boundary data), or coefficients which are not smooth, e.g.

$$-\Delta u = \mathsf{sgn}(\frac{1}{2} - |x|) \quad in \quad \Omega$$
$$u = 0 \quad in \quad \partial\Omega$$

  with $\Omega = (-1,1) \times (-1,1) \subset \mathbb{R}^2$
- multiply by a compact supported function ϕ and integrate by parts

$$\int_\Omega \nabla u \nabla \phi = \int_\Omega \mathsf{sgn}(\frac{1}{2} - |x|)\phi$$

- this expression makes sense even if u is not twice differentiable: we need to introduce appropriate functional spaces where the solutions of such equations live

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

Banach Spaces

- let $(E, ||.||_E)$ be a normed linear real vector space
- $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E if

$$\forall \epsilon \quad \exists M, \quad \text{such that} \quad \forall p, q \quad ||v_p - v_q||_E \leq \epsilon$$

- $(E, ||.||_E)$ is a Banach space if it is complete: all its cauchy sequences converge to an element in E
- Examples: $\mathbb{R}^N$, continuous function on a closed domain $C(\bar{\Omega})$...

Hilbert Spaces

- Let E be a linear real vector space having a scalar product $(\cdot, \cdot)_E$
- E is a Hilbert space if it is a Banach space with the norm defined by the scalar product, i.e. $||v||_E = (v, v)_E$
- Examples: $\mathbb{R}^N$, sequences $(v_n)_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} |v_n|^2 < \infty$...

Functional Spaces

- let $\Omega \subset \mathbb{R}^N$, some functional spaces:
  - $k-$times continuous differentiable functions:

$$C^k(\Omega) = \left\{ u \in C(\Omega) : \frac{\partial u}{\partial x_i} \in C(\Omega),\, 1 \leq i \leq N \right\}$$

  - functions with compact support

$$C_0^k(\Omega) = \left\{ u \in C^k(\Omega) : u \text{ has compact support in } \Omega \right\}$$

  - L^p lebesgue integrable functions:

$$L^p(\Omega) = \left\{ u : \int_\Omega |f|^p < \infty \right\}$$

- L^p for $1 \leq p \leq \infty$ are Banach spaces with norm $||f||_{L^p(\Omega)} = \left[\int_\Omega |f|^p \right]^{\frac{1}{p}}$. Also, $L^2(\Omega)$ is a Hilbert space with $(f,q)_{L^2(\Omega)} = \int_\Omega fg$

Sobolev Spaces

- Let $f \in L^1(\Omega)$, g is the weak partial derivative, $\frac{\partial f}{\partial x_i}$, of f if

$$\int_\Omega f \frac{\partial \phi}{\partial x_i} = -\int_\Omega g\phi \qquad \forall \phi \in C_0^\infty(\Omega).$$

  Remark: If f has classic derivatives they coincides with the weak.

- Sobolev spaces are defined

$$W^{1,p} := \left\{ u \in L^p(\Omega) : \frac{\partial u}{\partial x_i} \in L^p(\Omega), 1 \le i \le N \right\},$$

  the functions in L^p having weak derivatives in L^p.

- For the case $p = 2$ we denote $H^1(\Omega) := W^{1,2}(\Omega)$ is a Hilbert space with the scalar product

$$(f, g)_{H^1(\Omega)} = \int_\Omega fg + \int_\Omega \nabla f \nabla g$$

Sobolev Spaces

- We have convergence of a sequence of elements (u_m) in a Banach space E to an element $u_\infty \in E$ whenever

$$\lim_{m \to \infty} ||u_m - u_\infty||_E = 0$$

- $H_0^1(\Omega)$ is the set of all $u \in H^1(\Omega)$ such that u is the limit in $H^1(\Omega)$ of a sequence u_m, with $u_m \in C_0^\infty(\Omega)$.

Useful Inequalities

- **The Cauchy-Schwarz inequality:** Let u and v belong to $L^2(\Omega)$ then u, v $L^1(\Omega)$ and

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^2(\Omega)}||v||_{L^2(\Omega)}$$

- **(Poincaré-Friedrichs inequality)** Suppose that Ω is a bounded open set in $\mathbb{R}^n$ (with a sufficiently smooth boundary $\partial\Omega$) and let $u \in H_0^1(\Omega)$ then there exists a constant c_*, independent of u, such that

$$\int_\Omega |u(x)|^2 \, dx \leq c_* \sum_{i=1}^{n} \int_\Omega \left| \frac{\partial u}{\partial x_i}(x) \right|^2 \, dx$$

References

- U. Clarenz, U. Diewald, M. Rumpf: Anisotropic geometric diffusion in surface processing. IEEE Visualization 2000
- T. Preuer, M. Rumpf: A level set method for anisotropic geometric diffusion in 3D image processing. SIAM J. Applied Mathematics, 2002