Lecture 15

- Laplace-Beltrami Operator
- Diffusion on Surfaces
- Heat Method for Geodesic Distances

Riemannian Metric in Local Parametrisation

- Let $\sigma : D \to \sigma(D)$ be a local parametrisation of $S \subset \mathbb{R}^M$ around $\sigma(x) = p$
- Let g be the metric on S induced by the Euclidean space of $\mathbb{R}^M$. Then

$$g(v, w) = \sum_{i,j=1}^{n} g_{i,j} v_i w_j := \sum_{i,j=1}^{n} \left\langle \frac{\partial \sigma}{\partial x_i}, \frac{\partial \sigma}{\partial x_j} \right\rangle v_i w_j$$

for all $v = \sum v_i \partial_i \sigma$ and $w = \sum w_i \partial_i \sigma$.

The Intrinsic Gradient

For $f : S \to \mathbb{R}$: and $p \in S$,

- $\mathrm{D}f(p)(v) := \frac{d}{dt} f(\gamma(t))|_{t=0}$ for some $\gamma : I \to S$, s.t. $\gamma(0) = p, \gamma'(0) = v$, is well defined (does not depend on γ)
- There exists a unique element $\nabla_S f \in T_pS$ s.t.

$$g(\nabla_S f(p), v) = \mathrm{D}f(p)(v)$$

for all $v \in T_pS$. This intrinsec gradient is given by

$$\nabla_S f = \sum_{i=1}^{N} a_i \partial_i \sigma, \quad \text{with} \quad a_i = \sum_{j=1}^{N} g^{i,j} \frac{\partial (f \circ \sigma)}{\partial x_j}$$

with $(g^{i,j})_{1 \le i \le N}$ the inverse matrix of $(g_{i,j})_{1 \le i \le N}$.

The Intrinsic Divergence

- If f is defined over S and $\sigma : D \to S$ are local coordinates (local chart)

$$\int_{\sigma(D)} f \, dS = \int_D f \sqrt{|g|} \, dx$$

- We have that the intrinsec divergence of a vector field $V = \sum_{i=0}^N w_i \partial_i \sigma$

$$div_S V := \frac{1}{\sqrt{|\det g|}} \sum_{i=1}^N \frac{\partial}{\partial x_i} (w_i \sqrt{|\det g|})$$

satisfies

$$\int_S f \, div_S V \sqrt{|\det g|} dx = - \int_S g(\nabla_S f, V) \sqrt{|\det g|} \, dx$$

equivalently

$$\int_S f \, div_S V \, dS = - \int_S g(\nabla_S f, V) \, dS$$

for all f with compact support ($f \circ \sigma$ with compact support on D).

Laplace-Beltrami Operator

- The Laplace Beltrami operator of a function $f : S \to \mathbb{R}$ in local coordinates $\sigma : D \to S$, $D \subset \mathbb{R}^N$, is given by

$$\Delta_S f := div_S(\nabla_S f) = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\sqrt{|\det g|}\, g^{i,j} \frac{\partial f}{\partial x_j} \right)$$

Laplace-Beltrami Operator on Surfaces

Relation to Embedding Space $\mathbb{R}^3$

- Define f in $\mathbb{R}^3$ volume around S. The projection of $\mathbb{R}^3$ gradient $\nabla_3 f$ to tangential space $T_p S$ equals intrinsic gradient (Id is the identity matrix):

$$\nabla_S f = \left(I_d - \vec{n}\,\vec{n}^\top\right) \nabla_3 f$$

Local Coordinates

- We have that

$$\nabla_S f = D_\sigma(\mathbf{I}^{-1}\nabla(f \circ \sigma))$$

$$div_S V = \frac{1}{\sqrt{\det \mathbf{I}}} div(\sqrt{\det \mathbf{I}}(D\sigma)^+ V \circ \sigma)$$

where $D\sigma^+$ is the pseudoinverse of D_σ.

Example: Linear Diffusion on a Surface

- Consider surface $\sigma : D \to S \subset \mathbb{R}^3$ and a function (image data) U on $\sigma(D)$
- Linear diffusion (heat flow) equation

$$U_t = \Delta_S U$$

- This is a gradient descent for the energy

$$E[U] := \frac{1}{2} \int_S ||\nabla_S U||^2 dS$$

- General result on linear diffusion: On a compact Riemannian manifold, the heat equation for given initial data has a unique solution which satisfies a maximum-minimum principle.
  - Maximum-minimum principle: If at $t = 0$ one has $U_1 \leq U(\cdot, t) \leq U_2$, then $U_1 \leq U(\cdot, t) \leq U_2$ for all t
  - Compact domain: necessary for maximum-minimum principle – also satisfied for standard case of plain rectangular image with reflecting boundary conditions

Example: Linear Diffusion on a Surface

Left to right: Noisy image on surface and two stages of linear diffusion smoothing on the surface at progressive times (Bertalmo et al. 2001)

Implicit Surfaces

- **Main assumption:** Level set representation of the surface $S = \left\{x \in \mathbb{R}^3 : \psi(x) = 0\right\}$ and the data U is defined in a band surrounding it
- There are different techniques to arrive at an implicit representation from a triangulation:
  e.g. solving Hamilton Jacobi equation $||\nabla\psi|| = 1$
- Same for an extension of initial data:
  e.g. s.t. it is constant normal to each level set of ψ. Namely s.t. $\langle\nabla U, \nabla\psi\rangle = 0$.
  This can obtained by the steady state of

$$\frac{\partial U}{\partial t} + sign(\psi)\,\langle\nabla U, \nabla\psi\rangle = 0$$

Heat Flow on Implicit Surfaces

- Let S is the zero level set of ψ and U be defined on a neighborhood of S.
- Linear diffusion (heat flow) equation $U_t = \Delta_S U$ is a gradient descent for the energy

$$E[U] := \frac{1}{2} \int_S ||\nabla_S U||^2 dS.$$

We embed the energy E in $\mathbb{R}^3$ by defining

$$\tilde{E}[U] := \frac{1}{2} \int_{\Omega \subset \mathbb{R}^3} ||P_{\nabla \psi} \nabla U||^2 \delta(\psi) ||\nabla \psi|| dx,$$

where δ is the delta of Dirac and $P_{\vec{v}} = \left(I_d - \frac{\vec{v}\vec{v}^\top}{||\vec{v}||^2} \right)$, is the projection to the plane $\perp \vec{v}$.

- **Proposition:** The gradient descent of $\tilde{E}$ is given by

$$\frac{\partial U}{\partial t} = \frac{1}{||\nabla \psi||} div \left(P_{\nabla \psi} \nabla U ||\nabla \psi|| \right)$$

Heat Flow on Implicit Surfaces

- The flow is locally independent of the embedding function ψ
- If ψ is the signed distance function the gradient descent simplifies to

$$\frac{\partial U}{\partial t} = div(P_{\nabla \psi} \nabla U)$$

- Gradient descent of $\tilde{E}$ corresponds to computing the Laplace-Beltrami operator for S in implicit form

Nonlinear Diffusion on Surfaces

- Nonlinear isotropic diffusion in $\mathbb{R}^2$

$$U_t = div(g(||\nabla U_\rho||^2 \nabla U)$$

with nonnegative diffusivity function g applied to gradient of possibly presmoothed image $U_\rho = U * K_\rho$, with K_ρ Gaussian

- Example: TV flow

$$g(||\nabla U||^2) = \frac{1}{||\nabla U||}$$

- Translating divergence and gradient into the corresponding intrinsic expressions gives

$$U_t = div_S\left(\frac{\nabla_S U}{||\nabla_S U||}\right)$$

Nonlinear Diffusion on Surfaces: Implicit Surface

- Embedding of surface TV into $\mathbb{R}^3$ is given by

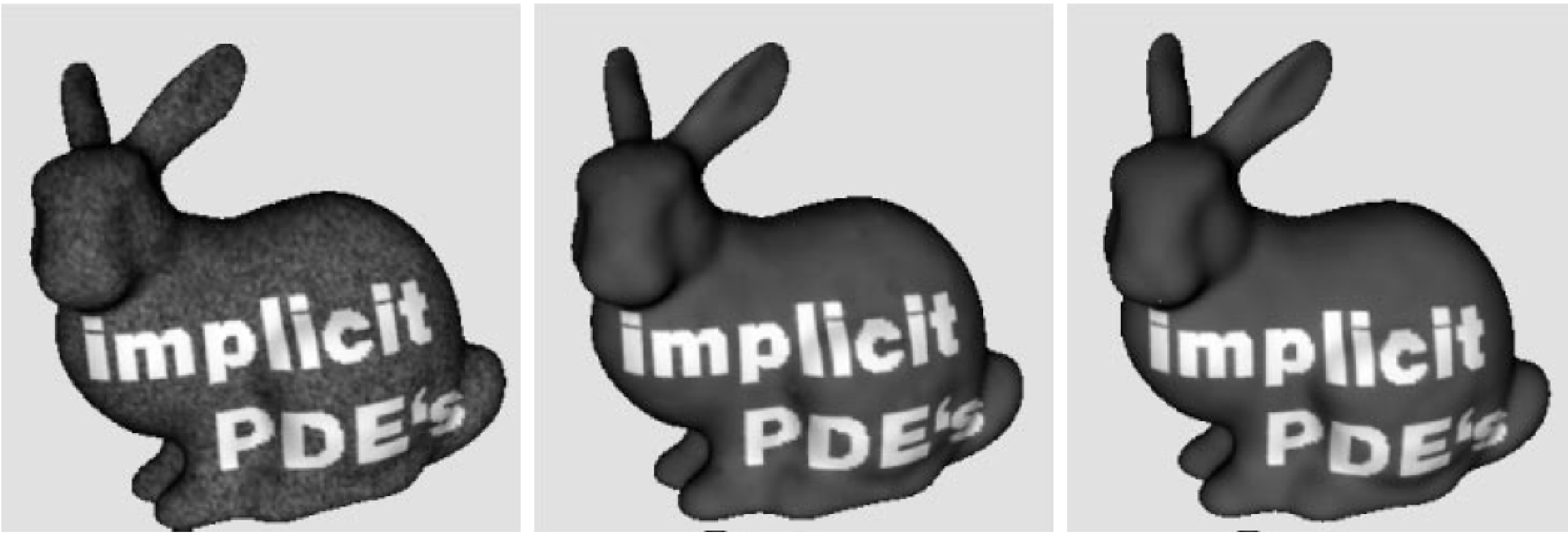
$$\frac{\partial U}{\partial t} = \frac{1}{||\nabla\psi||} div \left(\frac{P_{\nabla\psi}\nabla U}{||P_{\nabla\psi}\nabla U||} ||\nabla\psi|| \right)$$

- Can be obtained as the gradient descent of

$$\frac{1}{2} \int_{\Omega\subset\mathbb{R}^3} ||P_{\nabla\psi}\nabla U|| \delta(\psi) ||\nabla\psi|| dx$$

or by directly computing the PDE in implicit form

Nonlinear Diffusion on Surfaces



Left to right: Noisy image on surface and two stages of nonlinear isotropic diffusion smoothing on the surface (TV diffusivity) at progressive times (Bertalmo et al. 2001)

Diffusion of Directional Data on Surfaces

- Assume we have data of the form $U : S \to \mathbb{S}^2$. We want to minimize an energy of the form

$$\int_S ||\nabla_S U||^p dS.$$

- The gradient descent of this energy is given by the coupled system of PDEs

$$\frac{\partial U_i}{\partial t} = div_S \left(||\nabla_S U|\,|^{p-2} \nabla_S U_i \right) + U_i ||\nabla_S U||^p, \quad i = 1, 2$$

- Embedding S into the zero level set of ψ we obtain the gradient descent flows

$$\frac{\partial U_i}{\partial t} = \frac{1}{||\nabla \psi||} div \left(\frac{P_{\nabla \psi} \nabla U_i}{||P_{\nabla \psi} \nabla U||^{2-p}} ||\nabla \psi|| \right) + U_i ||P_{\nabla \psi} \nabla U||^p, \quad i = 1, 2.$$

Fundamental Solution of Linear Surface Diffusion

- Consider linear diffusion equation on surface

$$U_t = \Delta_S U, \quad U(t=0) = U_0$$

- For any $T > 0$, solution is given by

$$U(P,T) = \int_S U_0(Q) H(P,Q,T) dQ$$

- H is called fundamental solution or heat kernel for the surface diffusion
- Special case: In $\mathbb{R}^d$, the fundamental solution is given by $U(t=T) = G_{\sqrt{2T}} * U_0$

Short Time Solution of Linear Diffusion on Surfaces

- For small T, the fundamental solution of linear surface diffusion is approximated by

$$H(P,Q,T) \approx K(P,Q,T) := \frac{1}{4\pi T} \exp -\frac{\rho(P,Q)^2}{4T}$$

where $\rho(P,Q)$ is a smooth function with range $\mathbb{R} \cup \{+\infty\}$ such that

$$\begin{aligned} \rho(P,Q) &= d(P,Q), & d(P,Q) &\le \delta/2 \\ \rho(P,Q) &= +\infty, & d(P,Q) &\ge \delta \end{aligned}$$

and δ is the injectivity radius (where the exponential map is injective)

- Short-time kernel method for the computation of surface diffusion: Use K to approximate H

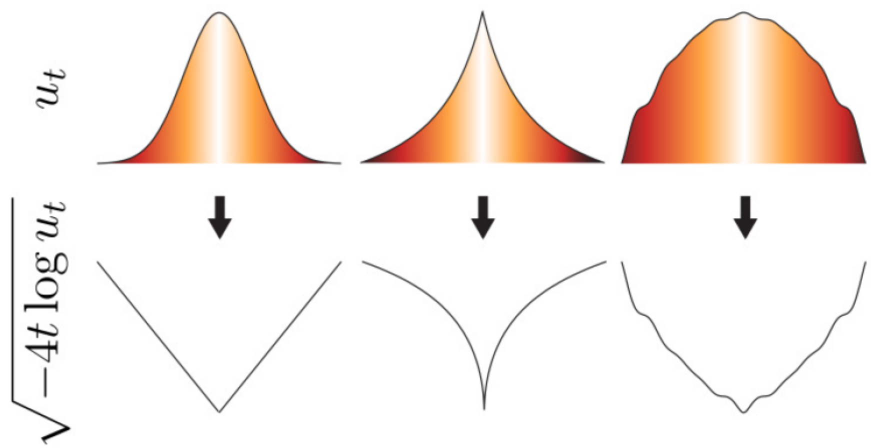
Remarks:

1. $K(P,Q,T)$ is a generalised and smoothly cut-off Gaussian on the surface

2. The true fundamental solutions H for all T can be computed via a series expansion whose first member is K

Distances on Surfaces

- Idea: use heat kernel to compute geodesic distances between points
- drawbacks: only accurate for small t, sensitive to noise



Heat kernel, here u_t, and compute distance function, (K. Crane et al. 2013)

Distances on Surfaces

Alternative solution (K. Crane et al. 2013)

- geodesics have constant speed
- magnitude of the heat kernel is irrelevant
- **Algorithm:**

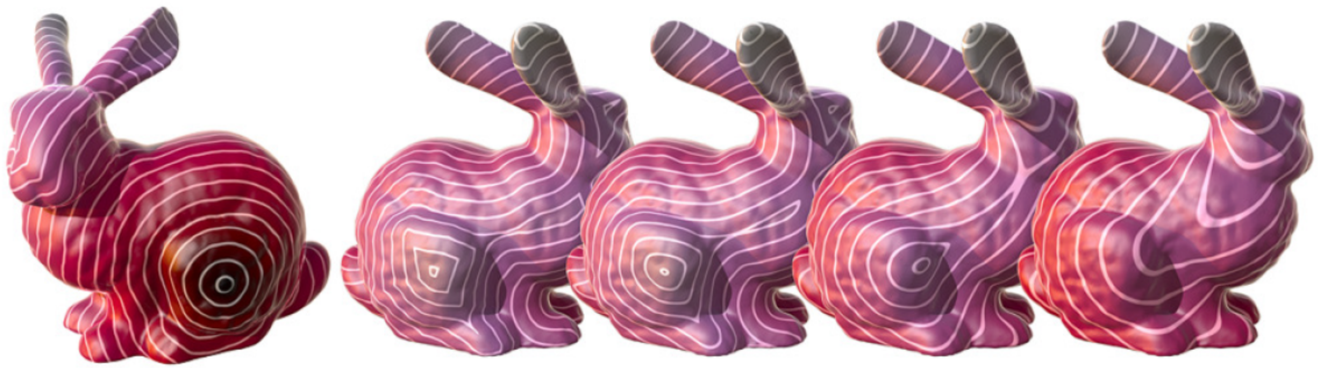
  1. Solve $U_t = \Delta_S U$, for fixed $t, U_0 = \delta_p$

  2. Compute $-\frac{\nabla U}{||\nabla U||}$

  3. Solve poisson equation $\Delta_S \Phi = divX$

  $\Phi(q)$ approximates the geodesic distance between p and q

Distances on Surfaces



Bunny mesh with increasing time parameter t, (K. Crane et al. 2013)

- Larger values for t lead to smoother, more inaccurate solutions

References

- K. Crane, C. Weischedel, M. Wardetzky: Geodesics in Heat: A New Approach to Computing Distance Based on Heat Flow. ACM Transactions on Graphics 32(5), 2013
- M. Bertalmío, L.-T. Cheng, S. Osher, G. Sapiro: Variational problems and partial differential equations on implicit surfaces. Journal of Computational Physics 174:759–780, 2001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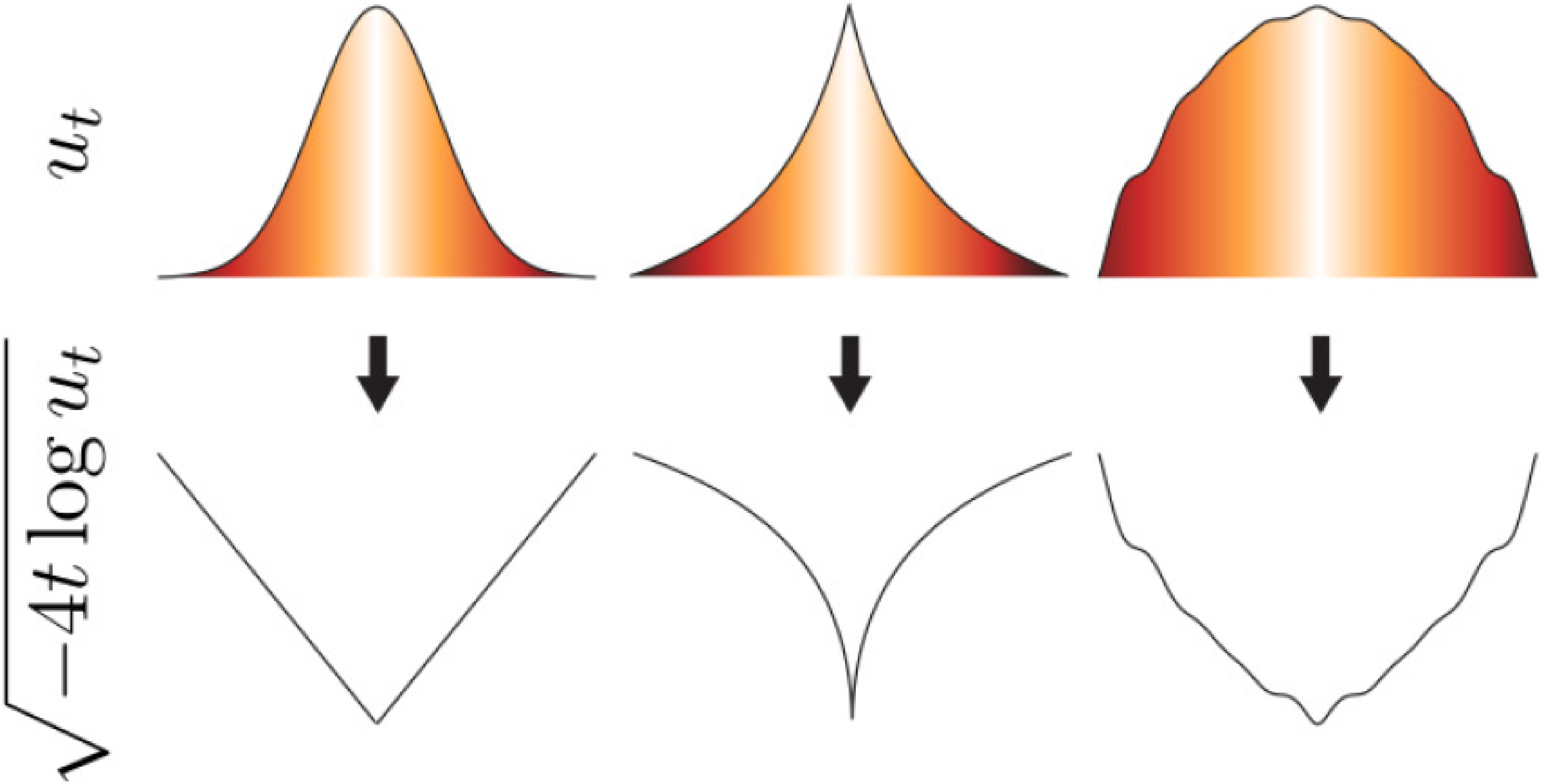

u_t
$\sqrt{-4t \log u_t}$