Lecture 14

- Geodesics and Curve Evolutions
- Hamilton Jacobi Equations
- Laplace-Beltrami Operator

Equal Distance Contour

- Given a source area $K \subset S$. We want to find a curve evolution s.t. the graph of $\alpha(\cdot, t)$ is $\{p \in S : d_s(p, K) = t\}$, the equal distance contour of distance t
- Consider the general evolution

$$\alpha_t = N \times \overrightarrow{t^{\alpha}}, \quad \alpha(u, 0) = \alpha_0(u)$$

- **Lemma:** The curve $\beta(t) := \alpha(u, t)|_{u=u_0}$ is a geodesic

Equal Distance Contours

- **Proposition:** The equal distance contour evolution of an initial curve u_0 is given by

$$\alpha_t = N \times \overrightarrow{t^{\alpha}} \quad \alpha(\cdot, 0) = u_0(\cdot)$$

where $\overrightarrow{t^{\alpha}}$ are the tangent unit vector of the equal distance contours $\alpha(\cdot, t)$

- Given a source area K we can find the equal distance contours $\{p \in S : d_s(p, K) = t\}$ choosing u_0 with graph equal to the boundary of K

- If source is a point choose K to be a small circle around the point

2D Projection

- Implementing directly an evolution of a 3D curve is quite cumbersome. We are interested is the projection Π of this 3D curve in the xy plane.
- **Proposition:** The projected equal distance contour evolution is given by

$$C_t = V_N \overrightarrow{n} \qquad c_0 = \partial\pi(K) \tag{1}$$

  where

$$V_N = \left\langle \overrightarrow{n}, \Pi(N \times \overrightarrow{t^{\alpha}}) \right\rangle = \sqrt{\frac{(1+q^2)n_1^2 + (1+p^2)n_2^2 - 2pqn_1n_2}{1+p^2+q^2}},$$

  with $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and $\overrightarrow{n} = (n_1, n_2)$
- This means that

$$V_N = \sqrt{an_1^2 + bn_2^2 - cn_1n_2},$$

  where a, b, c depend on the surface gradient and can be computed once at the start

Level Sets Propagation

- **Lemma**: Consider a function $u : U \subset \mathbb{R}^2 \to \mathbb{R}$ whose level sets correspond to equal distance contours. Then

$$\Pi(N \times \overrightarrow{t^\alpha}) = \frac{(u_x(1+q^2) - pqu_y, u_y(1+p^2) - pqu_x)}{\sqrt{(1+p^2+q^2)(u_x^2 + u_y^2 + (qu_x - pu_x)^2)}}$$

and the 2D curve evolution

$$\tilde{C}_t = -\Pi(N \times \overrightarrow{t^\alpha}) = -\frac{(u_x(1+q^2) - pqu_y, u_y(1+p^2) - pqu_x)}{\sqrt{(1+p^2+q^2)(u_x^2 + u_y^2 + (qu_x - pu_x)^2)}} \qquad (2)$$

can be used to compute geodesic paths

Consequences:

- The proposition in slide 4 follows using $\overrightarrow{n} = \frac{\nabla u}{|\nabla u|}$
- We can first compute u and then solve this evolution to compute the paths of geodesics

Hamilton Jacobi Equations

- Computing the evolution of the equal distance contours corresponds to solving

$$|\nabla u|^2 = V_N^2 = \frac{(1+q^2)u_x^2 + (1+p^2)u_y^2 - 2pqu_xu_y}{1+p^2+q^2}.$$

  with boundary condition given by $u = 0$ at the source ∂K, where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$

- In other words the Hamilton Jacobi equation with hamiltonian H given by

$$H(u_x, u_y) = (1+q^2)u_x^2 + (1+p^2)u_y^2 - 2pqu_xu_y - (1+p^2+q^2) = 0.$$

Hamilton Jacobi Equations

Summing up:

- We compute u as the solution of the Hamilton Jacobi equation of Slide 6
- Afterwards, we compute the path of a geodesic from a given point using the evolution (2) of Slide 5

Finding minimal paths

- Let $A \subset S$ and $\mathbb{M}_A(x,y) := d_S((x,y,z(x,y)),A)$

  Lemma: All minimal paths between $K, D \subset S$ are given by the set

  $$G := \{(x,y,z(x,y)) \; : \; \mathbb{M}_K(x,y) + \mathbb{M}_D(x,y) = g_m\}$$

  where $g_m = \min_{(x,y)}(\mathbb{M}_K + \mathbb{M}_D)$

- Let α_K, α_D denote distance contour evolutions starting from $\partial K, \partial D$ respectively.

  Lemma: The tangential points of $\alpha_K(u,t)$ and $\alpha_D(\tilde{u},t)$ for $\tilde{t}+t = g_m$ generate the minimal paths from p_1 to p_2. i.e. lie on a constant parameter $u = u_0(\tilde{u} = \tilde{u}_0)$ of the propagating curve $\alpha_K(u,t)(\alpha_D(\tilde{u},\tilde{t}))$

Riemannian Metric in Local Parametrisation

- Let $p \in S \subset \mathbb{R}^M$ and ϕ be a local parametrisation around p with $\phi(x) = p$
- Consider the basis of T_pS, $\{\partial_i := \frac{\partial \phi}{\partial x_i}(x) \mid 1 \leq i \leq N\}$
- Consider the Riemmanian metric g on S induced by the Euclidean space of $\mathbb{R}^M$ (The first fundamental form in the case of a surface: $M = 3, N = 2$).
- **Lemma:** At T_pS we have that

$$g(v, w) = \sum_{i,j=1}^{n} g_{i,j} v_i w_j$$

for all $v = \sum v_i \partial_i$ and $w = w_i \partial_i$. where

$$g_{i,j} = \left\langle \frac{\partial \phi}{\partial x_i}, \frac{\partial \phi}{\partial x_j} \right\rangle = \langle \partial_i, \partial_j \rangle$$

The Gradient Over a Surface

- In $\mathbb{R}^N$, we have $\langle \nabla f(x), v \rangle = Df(x)(v)$, for all $v \in \mathbb{R}^N$
- **Lemma:** For a function $f : S \to \mathbb{R}$:

  i) $Df(p)(v) := \frac{d}{dt} f(\gamma(t))|_{t=0}$ for some $\gamma : I \to S$, s.t. $\gamma(0) = p, \gamma'(0) = v$, is well defined (does not depend on γ)

  ii) There exists a unique element $\nabla f \in T_p S$ s.t.

$$g(\nabla f(p), v) = Df(p)(v)$$

for all $v \in T_p S$. It is given by

$$\nabla f = \sum_{i=1}^{N} a_i \partial_i, \quad \text{with} \quad a_i = \sum_{j=1}^{N} g^{i,j} \frac{\partial (f \circ \phi)}{\partial x_j}$$

with $(g^{i,j})_{1 \le i \le N}$ the inverse of $(g_{i,j})_{1 \le i \le N}$ and ϕ local parametrisation.

1
2
3
4
5
6
7
8
9
10
11
12

The Divergence of a Vector Field

- For a vector field in $\mathbb{R}^N$ and a function f with compact support (zero outside a compact set)

$$\int_{\mathbb{R}^N} f\, div(V) dx = -\int_{\mathbb{R}^N} \langle \nabla f, V \rangle\, dx \tag{3}$$

- If f is defined over S and $\phi : U \to S$ are local coordinates

$$\int_{\phi(U)} f\ dS = \int_U f \sqrt{|g|}\, dx$$

  - Thus (3) transforms in the case of a surface into

$$\int_S f\, div_\phi(V) \sqrt{|\det g|} dx = -\int_S g(\nabla_\phi f, V) \sqrt{|\det g|}\, dx$$

  for f with compact support, and we get

$$div_\phi(V) = \frac{1}{\sqrt{|\det g|}} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (V_i \sqrt{|\det g|})$$

Laplace-Beltrami Operator

- The Laplace Beltrami operator of a function $f : S \to \mathbb{R}$ in local coordinates $\phi : U \to S$, $U \subset \mathbb{R}^N$, is given by

$$\Delta_\phi f := div_\phi(\nabla_\phi f) = \frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\sqrt{|\det g|}\, g^{i,j} \, \frac{\partial f}{\partial x_j} \right)$$