Lecture 13

- Length Minimising Properties
- Computing Geodesics

Geodesic Polar Coordinates

- Choose in T_pS a system of polar coordinates with ρ the polar radius and $\theta, 0 < \theta < 2\pi$ the polar angle.
- Up to the half-line l corresponding to $\theta = 0$ the diffeomorphism $\exp_p$ defines a system of polar coordinates.
- For any $q \in V$ geodesic circles and radial geodesics correspond to the images of the circles $\rho = const$ and lines $\theta = const$
- **Lemma:** (Gauss lemma) Let $\sigma : U \setminus l \to V \setminus L$ be a system of geodesic polar coordinates. Then the first fundamental form satisfy $E(\rho, \theta) = 1$, $F(\rho, \theta) = 0$. Moreover $G > 0$ for $\rho \neq 0$.

Length Minimisation Property of Geodesics

- **Theorem** Let p be a point in S. There exists a neighborhood $W \subset S$ of p such that if $\gamma : [0, t_1] \to W$ is a parametrised geodesic with $\gamma(0) = p, \gamma(t_1) = q$, and $\alpha : [0, t_1] \to S$ be a parametrised curve joining p and q we have

$$L(\gamma) \leq L(\alpha).$$

Moreover, if $L(\alpha) = L(\gamma)$ then their graphs in S coincide

Topology concepts:

- $K \subset \mathbb{R}^N$ is a closed set if for any sequences of points $(x_k) \subset K$ which converge to a point x_0 it follows that $x_0 \in K$
- $O \subset \mathbb{R}^N$ is a open set if for any $z_0 \in O$ there is a $r > 0$ s.t. $\{z \in \mathbb{R}^N : ||z - z_0|| < r\} \subset O$
- $O \subset \mathbb{R}^N$ is open iff $O^c := \mathbb{R}^N \setminus O$ is closed
- A surface is compact if it is closed and bounded

Length Minimisation Property of Geodesics

- We denote the distance between two points p, q of the surface with

$$d_S(p,q) := \inf \{L(\gamma) : \gamma : [t_0, t_1] \to S, \gamma(t_0) = p, \gamma(t_1) = q\}$$

where the infimum is taken aver all differentiable curves joining p and q

- If $K \subset S$, we denote the minimal distance from p to K

$$d_S(p,K) := \inf \{d_S(p,q) : q \in K\}$$

Length Minimisation Property of Geodesics

- **Proposition**: On compact manifolds, there is a radius $\delta > 0$ such that the exponential map is injective at least within distance δ around each point of the manifold.

- **Theorem (Hopf-Rinow):** A connected, compact manifold is geodesically complete, i.e. any pair of points can be joined by a lenght minimal geodesic.

Surface Sampling

Surface sampling can be useful for example to:

- acquire discrete samples from a continuous surface
- reduce the number of samples of a given mesh
- seed evenly a set of points on a surface:
  - relevant in numerical analysis to have a good accuracy
  - display 3D models with a low number of polygons

Typically samples should be approximately at the same distance from each other.

- naive solution: consider a regular grid on the domain of a surface parametrisation ϕ
- performs poorly if ϕ introduces heavy geodesic distortions.

  Computation of geodesic distances is therefore a central tool

Surface Sampling

Instead consider:

- (ϵ-covering): $\{x_1, ..., x_n\} \subset S$ s.t.

$$\bigcup_i B_\epsilon(x_i) = S \quad \text{where} \quad B_\epsilon(x) := \{y : d_S(x, y) \leq \epsilon\}$$

- (ϵ-separated sampling): $\{x_1, ..., x_n\} \subset S$ s.t.

$$\max\left(d_S(x_i, x_j)\right) \leq \epsilon$$

Assume we have a way to compute the geodesic distance:

- The *farthest point sampling algorithm:* greedy strategy adding the point of largest distance to the current of set sampling points
- The farthest point sampling $\{x_1, x_2, ..., x_n\}$ is an ϵ-covering that is ϵ-separated for

$$\epsilon = \max_{i=1,\ldots,n} \min_{i=1,\ldots n} d_S(x_1, x_j)$$

Equal Distance Contour

- Given a source area $K \subset S$. We want to find a curve evolution s.t. the graph of $\alpha(\cdot, t)$ is $\{p \in S : d_s(p, K) = t\}$, the equal distance contour of distance t
- Consider the general evolution

$$\alpha_t = N \times \overrightarrow{t^{\alpha}}, \quad \alpha(u, 0) = \alpha_0(u)$$

- **Lemma:** The curve $\beta(t) := \alpha(u, t)|_{u=u_0}$ is a geodesic

Equal Distance Contour

- **Proposition:** The equal distance contour evolution of an initial curve u_0 is given by

$$\alpha_t = N \times \overrightarrow{t^{\alpha}} \quad \alpha(\cdot, 0) = u_0(\cdot)$$

- Given a source area K we can find the equal distance contours $\{p \in S : d_s(p, K) = t\}$ choosing u_0 with graph equal to the boundary of K
- If source is a point choose K to be a small circle around the point

Level Sets Propagation

- Implementing directly an evolution of a 3D curve is quite cumbersome. We are interested is the projection π of this 3D curve in the xy plane.
- **Proposition:** The projected equal distance contour evolution is given by

$$C_t = V_N \overrightarrow{n} \qquad c_0 = \partial\pi(K)$$

moreover

$$V_N = \left\langle \overrightarrow{n}, \pi(N \times \overrightarrow{t^\alpha}) \right\rangle = \sqrt{\frac{(1+q^2)n_1^2 + (1+p^2)n_2^2 - 2pqn_1n_2}{1+p^2+q^2}},$$

where $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}$ and $\overrightarrow{n} = (n_1, n_2)$

- This means that

$$V_N = \sqrt{an_1^2 + bn_2^2 - cn_1n_2},$$

where a, b, c depend on the surface gradient and can be computed once at the start

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Finding minimal paths

- Let $A \subset S$ and $\mathbb{M}_A(x, y) := d_S((x, y, z(x, y)), A)$
- **Lemma:** All minimal paths between $K, D \subset S$ are given by the set

  $$G := \{(x, y, z(x, y)) \ : \ \mathbb{M}_K(x, y) + \mathbb{M}_D(x, y) = g_m\}$$

  where $g_m = \min_{(x,y)}(\mathbb{M}_K + \mathbb{M}_D)$
- Let α_K, α_D denote distance contour evolutions starting from $\partial K, \partial D$ respectively.
  Lemma: The tangential points of $\alpha_K(u, t)$ and $\alpha_D(\tilde{u}, t)$ for $\tilde{t} + t = g_m$ generate the minimal paths from p_1 to p_2. i.e. lie on a constant parameter $u = u_0(\tilde{u} = \tilde{u}_0)$ of the propagating curve $\alpha_K(u, t)(\alpha_D(\tilde{u}, \tilde{t}))$

References

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