Lecture 12

- Geodesics and Geodesic Curvature
- Exponential Map
- Length Minimising Properties

Definition of Geodesic

- A nonconstant parametrised curve $\gamma : I \to S$ is said to be geodesic at $t \in I$ if

$$\frac{D\gamma'(t)}{dt} = 0.$$

  It is a parametrised geodesic if it is geodesic for all $t \in I$

- For a parametrised geodesic $||\gamma'(t)|| \neq 0$ is constant. The parameter t of a parametrised geodesic γ is thus proportional to the length of γ.
- On the plane only straight lines are geodesic

Definition of Geodesic

- A regular connected curve C in S is a geodesic if for each $p \in C$ the arc length parametrisation near p is a parametrised geodesic
- Coincides with saying that for the corresponding arc length parametrisation, $\alpha''(s)$ is normal to the tangent plane.

Examples

- The great circles of a sphere are geodesics
- Geodescis for the cylinder $\{x^2 + y^2 = 1\}$ are: straight lines of the cylinder, circles obtained by horizontal cuts or helixes

Geodesic curvature

Let w be a differentiable field of unit vectors along $\alpha : I \to S$, with S an oriented surface.

- Letting
$$\frac{Dw}{dt} = \lambda(N \times w(t))$$
the real number $\lambda(t)$ denoted by $\left[\frac{Dw}{dt}\right]$ is called the algebraic value of the covariant derivative of w at t
- The sign of $\left[\frac{Dw}{dt}\right]$ depends on the orientation of S and

$$\left[\frac{Dw}{dt}\right] = \left\langle \frac{dw}{dt}, N \times w \right\rangle$$

Geodesic Curvature

Let C be a regular curve contained on an oriented surface S and α be an arch length parametrisation near some $p \in S$.

- The geodesic curvature κ_g of C at p is the algebraic value of $\alpha'(s)$ at p $\kappa_g := \left[\frac{D\alpha'(s)}{ds}\right] = \langle \alpha'', N \times \alpha' \rangle$
- Geodesic curvature changes sign when we change the orientation of either C or M
- **Proposition:** We have that

$$\kappa^2 = \kappa_g^2 + \kappa_n^2,$$

  where $\kappa_n = \langle \alpha'', N \rangle$ and geodesics are characterised as curves whose geodesic curvature is zero

Rate of Change of Angle Between Unit Vector Fields

- Let v, w be two unit vector fields along $\alpha : I \to S$. Consider a differentiable vector field $\bar{v}$ s.t. $\{v(t), \bar{v}(t), N(t)\}$ is positively oriented and let

$$w(t) = a(t)v(t) + b(t)\bar{v}(t),$$

with a, b are differentiable and

$$a^2 + b^2 = 1.$$

- **Lemma:** Let a, b be as above and ϕ_0 be such that $a(t_0) = cos\phi_0, b(t_0) = sin\phi_0$. Then

$$\phi = \phi_0 + \int_{t_0}^{t} (ab' - ba')dt$$

is s.t. $cos\phi(t) = a(t), sin\phi(t) = b(t)$, for $t \in I$ and $\phi(t_0) = \phi_0$

- **Lemma:** We have

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \frac{d\phi}{dt}$$

Characterisation of Geodesic Curvature

- **Proposition:** Let C be a curve in the oriented surface S with arc length parametrisation $\alpha(s)$ and $v(s)$ be a parallel field along α, then

$$\kappa_g(s) = \left[\frac{D\alpha'(s)}{ds}\right] = \frac{d\phi}{ds}$$

  where $\phi(s)$ is a determination of the angle from v to α' in the orientation of S (Slide 6)

- In other words: the geodesic curvature is the rate of change of the angle that the tangent to the curve makes with a parallel direction along the curve

- In the case of a plane κ_g reduces to the usual curvature

Example: Curvature Motion on Surfaces

- Curvature Motion in $\mathbb{R}^2$

$$U_t = ||\nabla U|| div \left(\frac{\nabla U}{||\nabla U||} \right)$$

- Assume we define the gradient and divergence over a surface S (more on that next week). Then we can extend it to

$$U_t = ||\nabla_\sigma U|| div_\sigma \left(\frac{\nabla_\sigma U}{||\nabla_\sigma U||} \right)$$

- This is called geodesic curvature flow since it turns out that

$$div_\sigma \left(\frac{\nabla_\sigma U}{||\nabla_\sigma U||} \right) = -\kappa_\sigma$$

is the geodesic curvature of the level line given by $U = const$ on S

Example: Curvature Motion on Surfaces

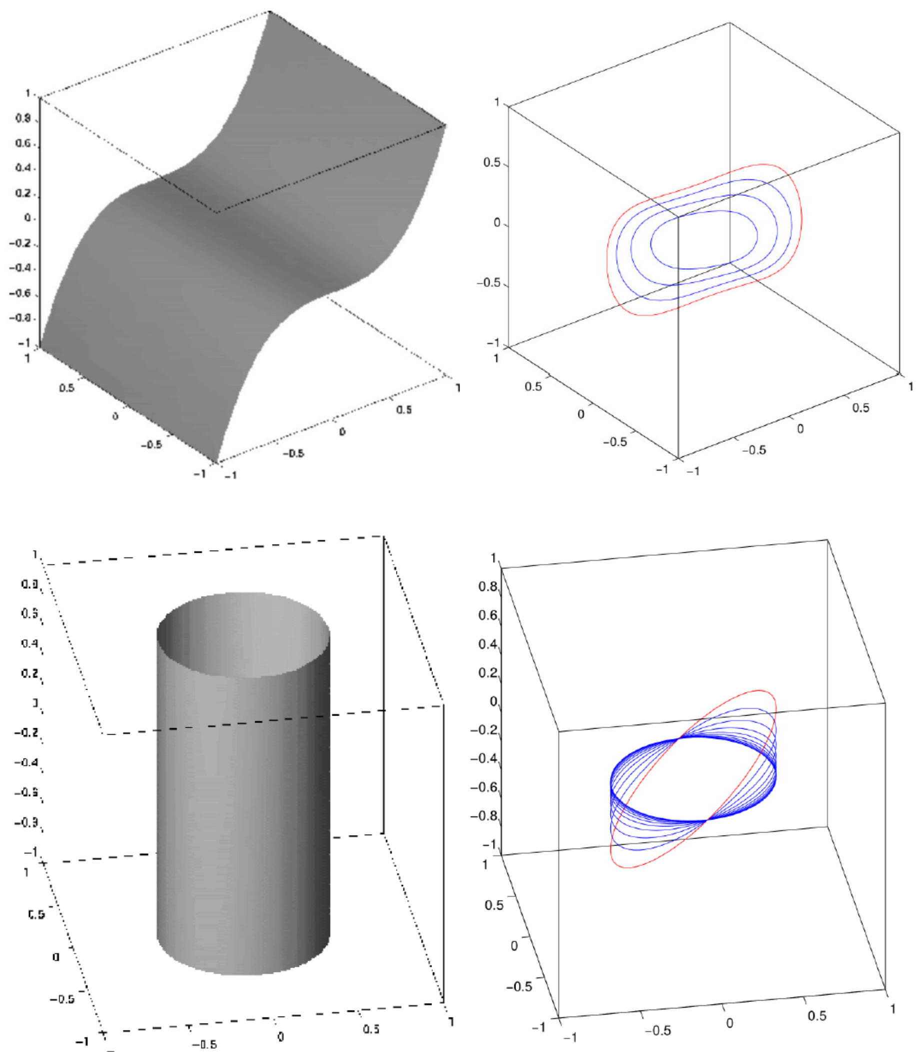


Figure: Four examples for geodesic curvature flow of level sets on surfaces. Evolution times are equally spaced. (Cheng et al. 2002)

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Equations in Local Coordinates

- **Lemma:** Let $\gamma : I \to S$ be a parametrised curve and let σ be a parametrisation of S around $\gamma(t_0)$. Let $\gamma(t) = \sigma(u(t), v(t))$ and the tangent vector $\gamma'(t)$ be given as

$$w = u'(t)\sigma_u + v'(t)\sigma_v.$$

Then w is parallel if and only if $u(t), v(t)$ solve (See Lecture 11 Slide 9)

$$u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0$$
$$v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0$$

As a consequence we obtain the existence of a local geodesic for any direction:

- **Proposition:** Given a point $p \in S$ and a vector $w \in T_p(S), w \neq 0$, there exists an $\epsilon > 0$ and a unique parametrised geodesic $\gamma : (-\epsilon, \epsilon) \to S$ such that $\gamma(0) = p, \gamma'(0) = w$

Exponential Map

- There always exists locally a unique parametrised geodesic with given direction (Slide 10). We denote the geodesic γ through $p = \gamma(0)$ on $v = \gamma'(0) \in T_pS$ with $\gamma(t, v)$
- **Lemma:** If the geodesic $\gamma(t, v)$ is defined for $t \in (-\epsilon, \epsilon)$, then the geodesic $\gamma(t, \lambda v)$, $\lambda \neq 0$, is defined for $t \in (-\epsilon/\gamma, \epsilon/\gamma)$ and $\gamma(t, \lambda v) = \gamma(\lambda t, v)$

  Since the speed of the geodesic is constant, we can go over its graph within a prescribed time by adjusting the speed appropriately
- If $v \in T_pS$ and $v \neq 0$ is such that $\gamma(|v|, v/|v|) = \gamma(1, v)$ is defined, we set

$$\exp_p(v) = \gamma(1, v) \quad \text{and} \quad \exp_p(0) = p$$

  the exponential map

  Corresponds to laying off a length equal to $||v||$ along the geodesic through p with direction of v.

Exponential Map

- **Proposition:** Given $p \in S$ there exists an $\epsilon > 0$ such that $\exp_p$ is defined and differentiable in the interior B_ϵ of a disk of radius ϵ of T_pS with center in the origin
- **Proposition:** $\exp_p : B_\epsilon \subset T_pS \to S$ is a diffeomorphism in a neighborhood U of the origin

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Normal and Geodesic Polar Coordinates

Geodesic Normal Coordinates

- Choose in the plane T_pS two orthogonal unit vectors e_1, e_1. Let $p \in U$ and V be s.t. $\exp_p : U \rightarrow V$ is a diffeomorphism.
- Any $q \in V$ can be written as $v = \exp_p(ue_1 + ve_2)$. We call u, v the normal coordinates of q
- The geodesics correspond to the image by $\exp_p$ of lines $u = at, v = bt$

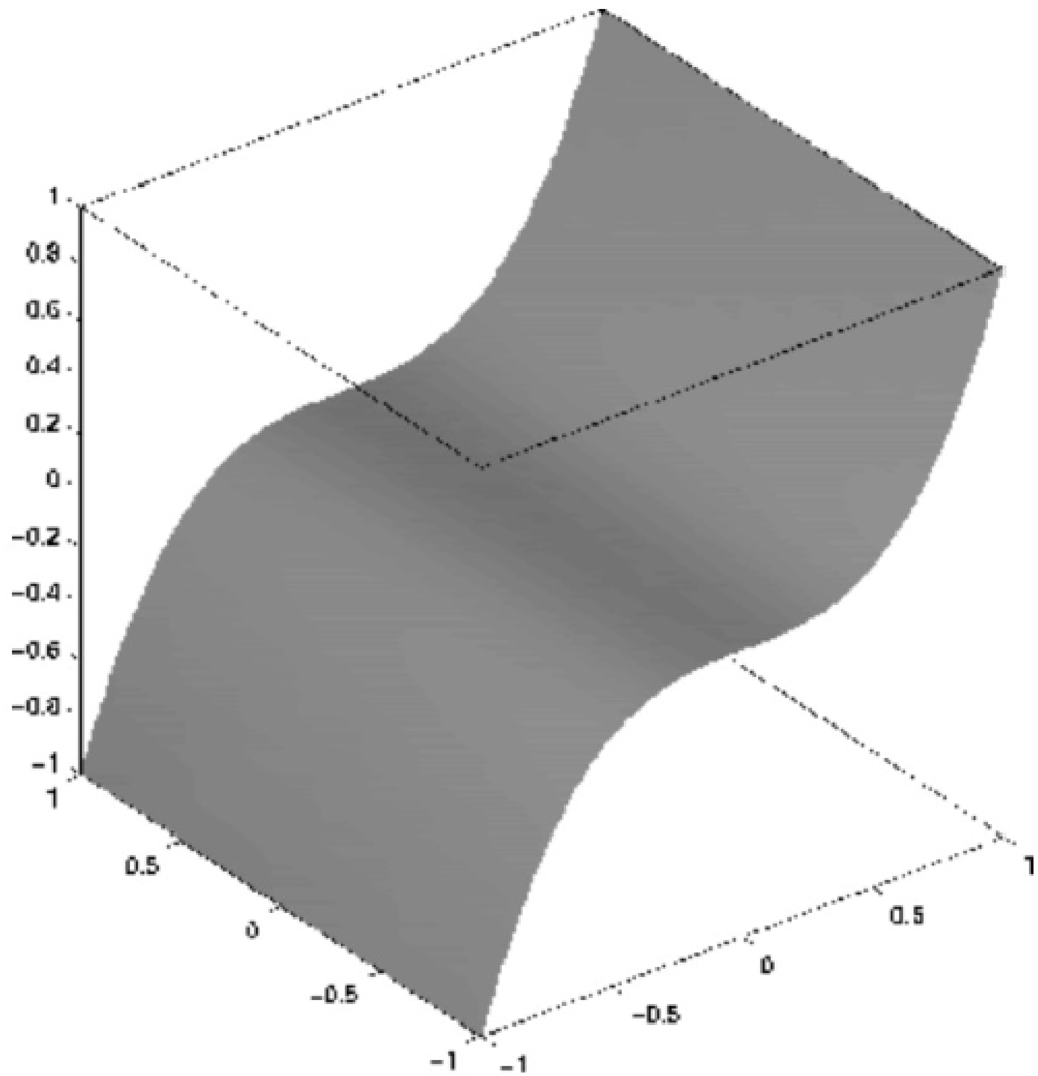
Geodesic Polar Coordinates

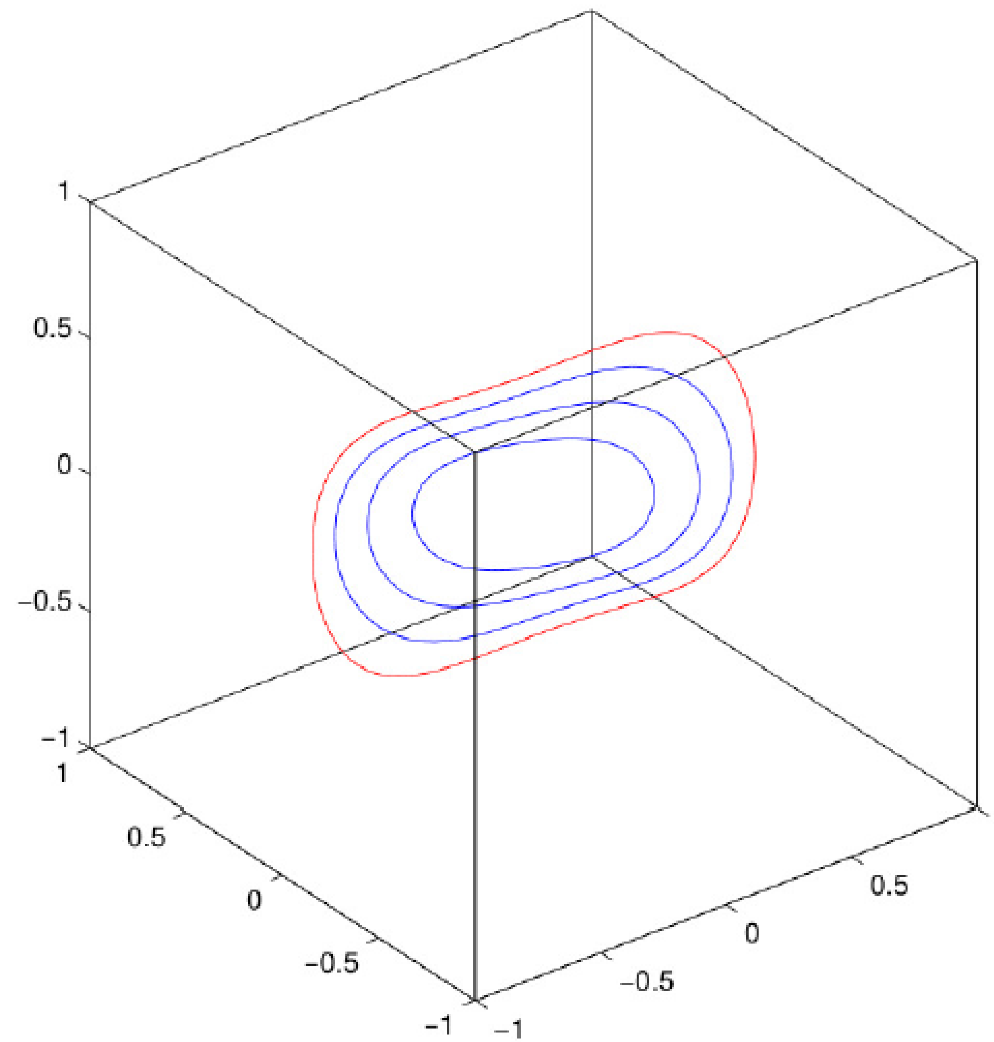
- Choose in T_pS a system of polar coordinates with ρ the polar radius and $\theta, 0 < \theta < 2\pi$ the polar angle.
- Up to the half-line l corresponding to $\theta = 0$ the diffeomorphism $\exp_p$ defines a system of polar coordinates.
- For any $q \in V$ geodesic circles and radial geodesics correspond to the images of the circles $\rho = const$ and lines $\theta = const$

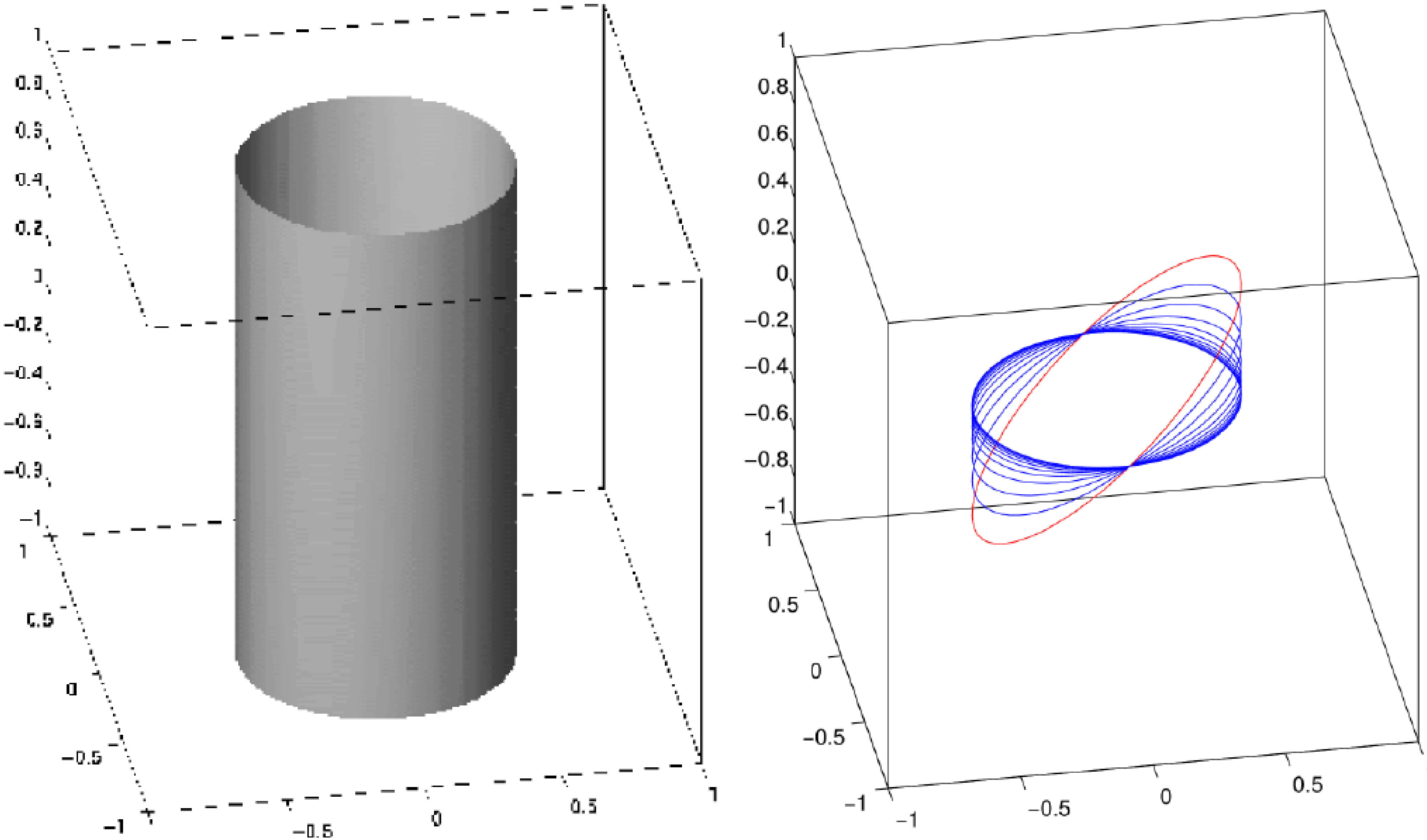
Length Minimisation Property of Geodesics

Theorem Let p be a point in S. Then, there exists a neighborhood $W \subset S$ of p such that if $\gamma : I \to W$ is a parametrised geodesic with $\gamma(0) = p, \gamma(t_1) = p, t_1 \in I$, and $\alpha : [0, t_1] \to S$ be a parametrised curve joining p and q we have

$$L(\gamma) \leq L(\alpha)$$


1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1


1
0.5
0
−0.5
−1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1



1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1