Lecture 11

- Vector Fields
- Covariant Derivatives
- Parallel Transport
- Geodesics

Vector Fields in $\mathbb{R}^2$

- A vector field in an open $U \subset \mathbb{R}^2$ is a map $w(q) = (a(q), b(q)) \in \mathbb{R}^2, \forall q \in U$
- It is differentiable if its coordinates a and b are
- Corresponds to assigning to each q a vector in $T_q\mathbb{R}^2$
- Example: The gradient ∇f of a smooth function $f : U \to \mathbb{R}$

Remark: In what follows we only consider differentiable vector fields

Trajectory of a Vector Field in $\mathbb{R}^2$

- Given a vector field w on $U \subset \mathbb{R}^2$, a trajectory of this field is a curve $\alpha(t) = (x(t), y(t))$, $t \in I$, such that

$$\alpha'(t) = w(\alpha(t))$$

- Writing $\alpha(t) = (x(t), y(t))$, the vector field w determines an ODE

$$\frac{dx}{dt} = a(x, y)$$
$$\frac{dy}{dt} = b(x, y)$$

solved by the curve α

Existence and Uniqueness of Trajectories

- Let w be a vector field in an open set $U \subset \mathbb{R}^2$ and let $p \in U$. From fundamental results about ODE's we have that locally:
  - **Theorem:** There exists a unique trajectory $\alpha : I \to U$ with $\alpha(0) = p$.
  - **Theorem:** There exists a neighborhood $V \subset U$ of p, an interval I, and a differentiable map $\alpha : V \times I \to U$, (the local flow of w at p) such that

$$\alpha(q, 0) = q, \qquad \frac{\partial \alpha}{\partial t}(q, t) = w(\alpha(q, t)).$$

- As a consequence We obtain the following:
  - **Lemma:** For w and p as above we obtain that if $w(p) \neq 0$, there exists $W \subset U$ of p and a differentiable $f : W \to \mathbb{R}$ such that f is constant along each trajectory of w and $\nabla f(q) \neq 0$ for all $q \in W$
  - This function f is called a local first integal of w

Vector Fields on a Surface

- A vector field w in an open set U of a regular surface S is a map $w(p) \in T_pS$ for each $p \in U$
- It is differentiable at p if for some local parametrisation $\sigma(u,v)$, the functions $a(u,v)$ and $b(u,v)$ given by

$$w(p) = a(u,v)\sigma_u + b(u,v)\sigma_v$$

  are differentiable at p (this definition does not depend on the choice of σ)
- Trajectories of w can be defined similarly as in the case of $\mathbb{R}^2$
- Previous properties about trajectories extend to surfaces (in particular results about existence of a *local flow* and *local first integral*)

Examples

- The vector field obtained by parametrising the meridians of the torus in arc length and taking $w(p)$ to be the corresponding tangent vector
- Similar procedure for the semimeridians of a sphere gives a vector field on $\mathbb{S}^2$ minus the poles
- Reparametrise all semimeridians of a sphere with same parameter $-1 < t < 1$ and define $v(p) = (1 - t^2)w(p)$ for points different than the poles and $v = 0$ at the poles
- There is no differentiable vector field w over all $\mathbb{S}^2$ such that $||w|| > 0$ (hairy ball theorem)

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Vector Fields on a Surface

Let w_1, w_2 be vector fields in an open U of the regular surface S which are linearly independent at some $p \in U$.

- **Theorem:** There exists parametrisation of a neighborhood of p, $V \subset U$, such that the coordinate lines of this parametrisation passing through q are tangent to the lines spanned by $w_1(q), w_2(q)$
- **Corollary:** For all $p \in S$ there exists a parametrisation $\sigma(u, v)$ such that the curves $u = const, v = const$, intersect orthogonally for each $q \in V$

  Such a σ is called orthogonal parametrisation.

Covariant Derivative

- Let w be a vector field on U, $p \in U$, and $v \in T_pS$
- Let α be a be a curve s.t. $\alpha(0) = p$ and $\alpha'(0) = v$ and consider w restricted to α
- The covariant derivative of w relative to v at p is the normal projection of $\frac{dw}{dt}(0)$ onto T_pS. It is denoted with $\frac{Dw}{dt}(0)$ or $D_v w(p)$

This definition does not depend on the choice of α. In order to prove this we make use of the Christoffel symbols Γ^k_{ij}. If σ is a local parametrisation, they are defined by:

$$\begin{aligned}
\sigma_{uu} =& \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_u + L_1 N, \\
\sigma_{uv} =& \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_u + L_2 N, \\
\sigma_{vu} =& \Gamma^1_{21}\sigma_u + \Gamma^2_{21}\sigma_u + \bar{L}_2 N, \\
\sigma_{vv} =& \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_u + L_3 N
\end{aligned}$$

Remark: All geometric concepts and properties expresses in terms of the Christoffel symbols are invariant under isometries (in particular Euclidean transformations).

Covariant Derivative

The definition of covariant derivative does not depend on the choice of α. In fact letting

$$w(t) = a(u(t), v(t))\sigma_u + b(u(t), v(t))\sigma_v = a(t)\sigma_u + b(t)\sigma_v$$

we get that

$$\begin{aligned} \frac{Dw}{dt} =& (a' + \Gamma^1_{11}au' + \Gamma^1_{12}av' + \Gamma^1_{12}bu' + \Gamma^1_{22}bv')\sigma_u + \\ & (b' + \Gamma^2_{11}au' + \Gamma^2_{12}av' + \Gamma^2_{12}bu' + \Gamma^2_{22}bv')\sigma_v. \end{aligned} \tag{1}$$

There is no explicit dependence on α

Covariant Derivative

- Let $\alpha : I \to S$ be a curve in the surface S. A vector field w along α is a map $w(t) \in T_{\alpha(t)}S$, for each $t \in I$. It is differentiable if that is the case for its components in some local parametrisation.

- If w is a differentiable vector field along $\alpha : I \to S$. the covariant derivative of w at t, $\frac{Dw}{dt}$, is well defined for all $t \in I$. It given by (eq. (1) slide 9)

$$
\begin{aligned}
\frac{Dw}{dt} =& (a' + \Gamma^1_{11} au' + \Gamma^1_{12} av' + \gamma^1_{12} bu' + \Gamma^1_{22} bv')\sigma_u + \\
& (b' + \Gamma^2_{11} au' + \Gamma^2_{12} av' + \Gamma^2_{12} bu' + \Gamma^2_{22} bv')\sigma_v
\end{aligned}
$$

- From the point of view external to the surface it corresponds to the projection of $\frac{dw}{dt}(t)$ onto the tangent plane $T_{\alpha(t)}S$.
  Intutitively $\frac{D\alpha'}{dt}$ is the acceleration of the point $\alpha(t)$ "as seen from the surface"

Remark: All geometric concepts and properties expresses in terms of the Christoffel symbols are invariant under isometries (in particular under Euclidean transformations).

1 2 3 4 5 6 7 8 9 10 11 12 13 14

Parallel Vector Field along a Curve

- A vector field w along a curve $\alpha : I \to S$ is said to be parallel if $\frac{Dw}{dt} = 0$ for every $t \in I$
- **Lemma:** If w and $\tilde{w}$ are parallel fields along α, then $\langle w(t), \tilde{w}(t)\rangle$ is constant. In particular $||w(t)||, ||\tilde{w}(t)||$ and the angle they form are constant

Parallel Transport

Let $\alpha : I \to S$ be a curve in S and let $w_0 \in T_{\alpha(t_0)}S, t_0 \in I$, then:

- **Theorem:** There exists a unique parallel vector field $w(t)$ along $\alpha(t)$ with $w(t_0) = w_0$
- The vector $w(t_1), t_1 \in I$ is called the parallel transport of w_0 along α at point t_1
- If α is regular, the parallel transport does not depend on the parametrisation of α

Parallel Transport

Examples:

- If S is a plane parallel transport corresponds to a constant vector along α
- The tangent vector field of a meridian of a sphere parametrised by arc length

Definition of Geodesic

- A nonconstant parametrised curve $\gamma : I \to S$ is said to be geodesic at $t \in I$ if

$$\frac{D\gamma'(t)}{dt} = 0.$$

  It is a parametrised geodesic if it is geodesic for all $t \in I$

- We have that for a parametrised geodesic $||\gamma'(t)||$ is constant and nonzero. Thus we may introduce the arc length $s = ct$ as a parameter. The parameter t of a parametrised geodesic γ is thus proportional to the length of γ.

- On the plane only straight lines are geodesic at each point