Lecture 10:

- Role of Gaussian Curvature
- Surface evolutions in Euclidean Space
- Curvature motion processes for surfaces

Surface Evolutions in $\mathbb{R}^d$

- Consider surfaces parametrised by connected domain $D \subset \mathbb{R}^2$
- Introduce additional time parameter $t \in [0, T]$, $T \geq 0$
- **Surface evolution:** differentiable function $\sigma : D \times [0, T] \to \mathbb{R}^d$
  - For each fixed t, $\sigma(\cdot, t)$ is a surface
  - Initial surface: $\sigma_0(\mathbf{u}) = \sigma(\mathbf{u}, 0)$
  - For each fixed $\mathbf{u} \in D$, $\sigma(\mathbf{u}, \cdot)$ is the "trajectory" of a surface point
  - Time derivative σ_t is called *surface flow*

Decompositions of Surface Evolutions in $\mathbb{R}^3$

Consider now a surface evolution in $\mathbb{R}^3$ with Euclidean metric

- Surface flow can be written as

$$\frac{\partial \sigma(\mathbf{u},t)}{\partial t} = \alpha_1(\mathbf{u},t)\vec{t}_1(\mathbf{u},t) + \alpha_2(\mathbf{u},t)\vec{t}_2(\mathbf{u},t) + \beta(\mathbf{u},t)\vec{n}(\mathbf{u},t)$$

- Assume that $\beta(\mathbf{u},t) = \tilde{\beta}(\sigma(\mathbf{u},t),t)$ depends on the surface *graphs* $\sigma(\mathbf{u},t)$ and t only (but not explicitly on the surface *parametrisation* $\mathbf{u}$)

- Similarly as for curve evolutions, one has that the evolution

$$\frac{\partial \tilde{\sigma}(\mathbf{u},t)}{\partial t} = \tilde{\beta}(\tilde{\sigma}(\mathbf{u},t),t)\,\vec{n}(\tilde{\sigma}(\mathbf{u},t),t)$$

  describes the same family of surface images, i.e. $\tilde{\sigma}(\cdot,t)$ is a reparametrisation of $\sigma(\cdot,t)$ for each t

- The shape of a surface σ is not changed by a flow $\sigma_t \perp \vec{n}$ (apart from cut-off at the boundary)
  The normal flow is what governs the shape evolution

- Proof similarly as for curves

Evolution of Level Surfaces

- Consider **3D image evolution:**
  smooth function $U : E \times [0,T] \to \mathbb{R}$ for 3D domain E
- Let **surface evolution** $\sigma : D \times [0,T] \to \mathbb{R}^3$ describe zero-level surface of U at each $t \in [0,T]$
- *Orientation convention:* The surface σ is oriented such that the surface normal $\vec{n}$ points to the region with smaller values of U, i.e.

$$\vec{n} = -\frac{\nabla U}{\|\nabla U\|}$$

- Vice versa, an image (evolution) can be defined for a given surface (evolution) by signed distance functions analogous to the 2D case

Evolution of Level Surfaces, cont.

- **Image flow**

$$\frac{\partial U}{\partial t} = \beta(x,y,z,t) \cdot \|\nabla U\| \ , \qquad\qquad (x,y,z) \in E, \quad t \in [0,T]$$

corresponds to **surface flow**

$$\frac{\partial \sigma}{\partial t} = \beta(\sigma(u,v,t),t) \cdot \vec{n}(u,v,t) \ , \qquad\qquad (u,v) \in D, \quad t \in [0,T]$$

- **Proof:** Analogous to curve evolution in 2D

Dilation and Erosion Flows in 3D

- Simplest example: $\beta = \mathrm{const}$
- Surface evolution:

$$\sigma_t = \pm \vec{n}$$

  ($-$: dilation, $+$: erosion)

- Image evolution:

$$U_t = \pm \|\nabla U\|$$

  ($+$: dilation, $-$: erosion)

- Properties of dilation and erosion are similar to the 2D case

Possible Curvature Motions in 3D

- **Goal:** Generalise curvature flow to surfaces in 3D
  - Application: e.g., smoothing of laser-scanned 3D surface data
- **Difficulty:** two principal curvatures (instead of a single curvature in 2D)
- **Consequence:** Several possibilities for surface evolutions corresponding to Euclidean curvature flow
- Candidates based on Euclidean invariants:
  - Mean curvature $\mathbf{H} = \frac{1}{2}(\kappa_1 + \kappa_2)$
  - Gauss curvature $\mathbf{K} = \kappa_1 \kappa_2$

Mean Curvature Motion in 3D

- Use $\beta = 2\mathbf{H} \ \ (= \kappa_1 + \kappa_2)$, then we have the surface evolution

$$\sigma_t = 2\mathbf{H}\,\vec{n}$$

  equivalent to the image evolution

$$U_t = 2\mathbf{H}\ \|\nabla U\|$$

- Image evolution can be rewritten into

$$U_t = \|\nabla U\|\ \mathrm{div}\,\frac{\nabla U}{\|\nabla U\|}$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Mean Curvature Motion in 3D, Equivalences

- Equivalent description of the image evolution as **smoothing along level sets**

$$U_t = U_{\xi\xi} + U_{\eta\eta}$$

where $\xi(x,y,z), \eta(x,y,z) \perp \nabla U(x,y,z)$ are orthogonal unit vectors, $\xi(x,y,z) \perp \eta(x,y,z)$, for each $x,y,z \in E$

- Equivalent reformulation of surface evolution as **smoothing of surface coordinates**

$$\sigma_t = \sigma_{uu} + \sigma_{vv}$$

if σ_u, σ_v are unit vectors and $\sigma_u \perp \sigma_v$

- Equivalent variational description: **gradient descent for surface area**

$$E[\sigma] = \iint_{\sigma} \sqrt{\det \mathbf{I}_{(u,v)}}\, \mathrm{d}u\, \mathrm{d}v$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Properties of 3D Mean Curvature Motion

Mean curvature motion as surface evolution:

$$\sigma_t = 2\mathbf{H}\,\vec{n}\ ,$$

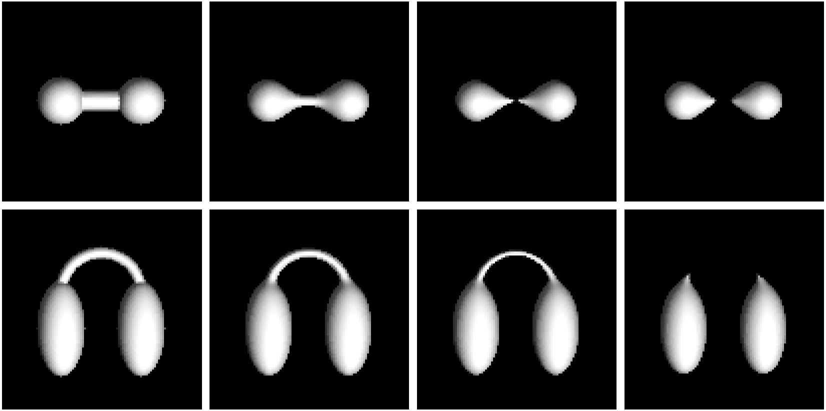
- Convex shapes shrink and evolve into "spherical" points
- Non-convex shapes do **not** necessarily stay connected
- Evolutions of non-convex shapes can include singularities
- Inclusions of surfaces are preserved

Mean curvature motion as image evolution:

$$U_t = U_{\xi\xi} + U_{\eta\eta}\ , \qquad \xi \perp \eta; \quad \xi, \eta \perp \nabla U$$

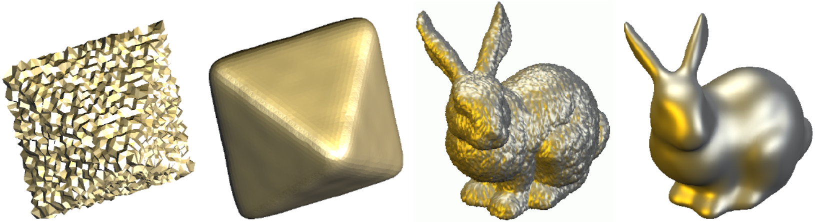
- Mean curvature motion is morphologically invariant

3D Mean Curvature Motion Examples



Top, left to right: "Dumbbell" surface and three stages of its evolution by mean curvature motion at progressive times ($t = 5$, $t = 7$, $t = 8$). The non-convex shape decomposes and develops singularities. **Bottom:** Same for "bent dumbbell" surface ($t = 1$, $t = 2$, $t = 3$) *(after Caselles, Sbert 1996)*

Surface Denoising by 3D Mean Curvature Motion



Left to right: Noisy octahedron – smoothed by mean curvature motion – noisy "Stanford bunny" – smoothed by mean curvature motion *(Clarenz, Diewald, Rumpf 2000)*

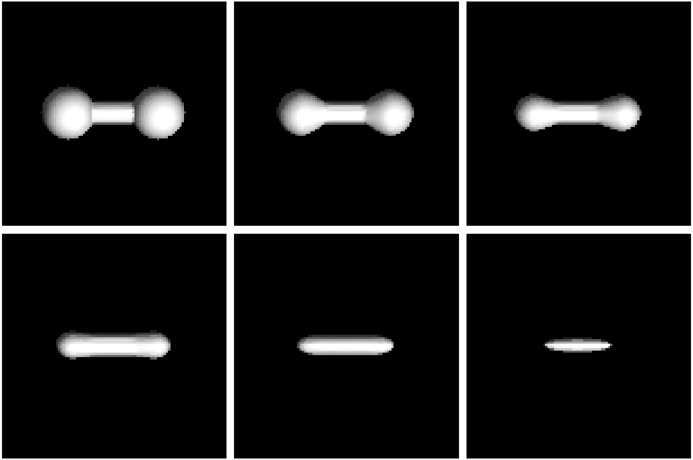
An Affine Invariant 3D Curvature Motion

- Based on Gaussian curvature $\mathbf{K}$
- Let $[\mathbf{K}]_+ := \max(\mathbf{K}, 0)$ and

$$\sigma_t = \operatorname{sgn}(\kappa_1) \cdot \left([\mathbf{K}]_+\right)^{1/4} \vec{n}$$

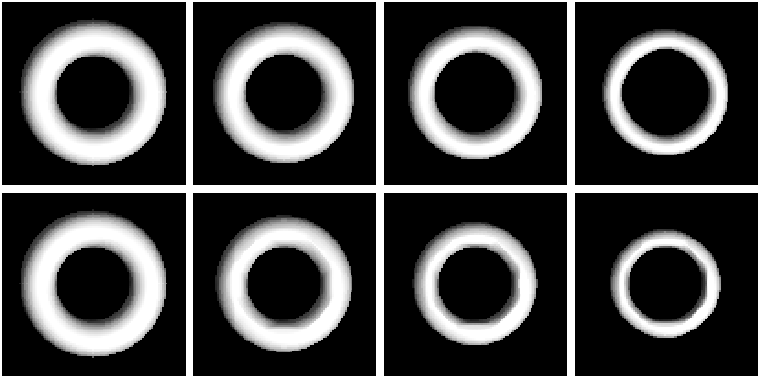
- Note that locations with negative Gaussian curvature do not move at all
- This flow is affine invariant
- It avoids singularities in a number of cases such as the dumbbell
- However, singularities can still occur

An Affine Invariant 3D Curvature Motion – Examples



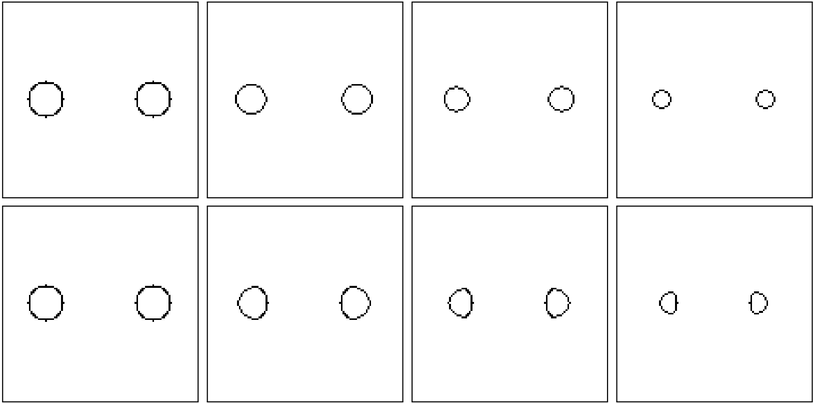
Top left to bottom right in rows: "Dumbbell" surface and five stages ($t = 5$, $t = 10$, $t = 15$, $t = 20$, $t = 25$) of its evolution by $\sigma_t = \mathrm{sgn}(\kappa_1) \cdot ([\mathbf{K}]_+)^{1/4}\, \vec{n}$. The non-convex shape stays connected and develops into a convex shape *(after Caselles, Sbert 1996)*

Comparison of 3D MCM and Affine Invariant Curvature Motion



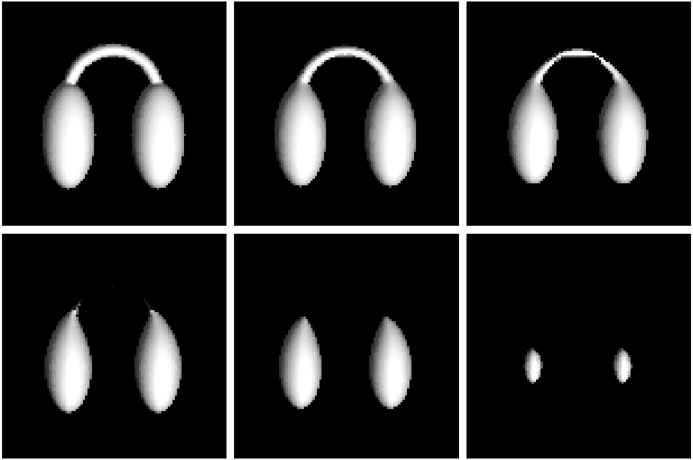
Top, left to right: Torus surface and three stages of its evolution under mean curvature motion ($t = 7$, $t = 14$, $t = 21$). **Bottom, left to right:** Torus surface and three stages of its evolution under $\sigma_t = \mathrm{sgn}(\kappa_1) \cdot ([\mathbf{K}]_+)^{1/4}\, \vec{n}$ ($t = 10$, $t = 20$, $t = 30$). Note that the inner equator does not move (as the Gaussian curvature remains always negative here) *(after Caselles, Sbert 1996)*

Comparison of 3D MCM and Affine Invariant Curvature Motion



Top, left to right: Cross-section of the torus surface from the previous slide and three stages of its evolution under mean curvature motion ($t = 7$, $t = 14$, $t = 21$). **Bottom, left to right:** Same for the affine invariant curvature motion $\sigma_t = \mathrm{sgn}(\kappa_1) \cdot ([\mathbf{K}]_+)^{1/4}\, \vec{n}$ ($t = 10$, $t = 20$, $t = 30$).

An Affine Invariant 3D Curvature Motion – Examples



Top left to bottom right in rows: "Bent dumbbell" surface and five stages of its evolution by $\sigma_t = \mathrm{sgn}(\kappa_1) \cdot ([\mathbf{K}]_+)^{1/4}\, \vec{n}$ ($t = 2$, $t = 4$, $t = 7$, $t = 10$, $t = 25$). This non-convex shape decomposes and develops singularities. *(after Caselles, Sbert 1996)*

Singularity–Free 3D Curvature Motion (Approach)

- A singularity-free curvature dependent motion can be based on minimising

$$E[\sigma] := \iint_{\sigma} \mathbf{K}^2 \; \mathrm{d}\sigma(u,v)$$

- Leads to higher order differential equation

References

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