Lecture 9

- Second Fundamental Form
- Curvature of a Surface
- Examples
- Gauss Egregium Theorem

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Shape Operator

Let M by an oriented regular surface having Gaussian map N.

- A differentiable map $N : M \to \mathbb{S}^2$ is said to be a Gauss map for M if $N(p) \perp T_pM$, for each $p \in M$
- We can identify $T_pM \cong T_{N(p)}\mathbb{S}^2$
- The shape operator $S_p : T_pM \to T_pM$ is the linear map given by $S_p(\boldsymbol{v}) = -\mathrm{D}N(p)(\boldsymbol{v})$, for all $\boldsymbol{v} \in T_pM$.

Claim: Let M be a path connected, oriented regular surface with Gaussian map. Then S_p vanishes for all $p \in M$ if and only if M is contained in a plane.

Shape Operator

- The second fundamental form of M at p, $\mathbf{II_p} : T_pM \times T_pM \to \mathbb{R}$ is given by

$$\mathbf{II_p}(\boldsymbol{v}, \boldsymbol{w}) = \langle S_p(\boldsymbol{v}), \boldsymbol{w} \rangle \,, \quad \boldsymbol{v}, \boldsymbol{w} \in T_pM$$

- S_p is a symmetric bilinear map. Therefore, so is $\mathbf{II_p}$

Local parametrisation

Let $\phi : U \to \phi(U)$ be a local parametrisation of an oriented regular surface M.

- In local coordinates (i.e. w.r.t. the basis $\phi_u, \phi_v,$) $S_p = S_{\phi(\boldsymbol{u})}$, is given by $S_{\boldsymbol{u}}(\boldsymbol{w}) = A\boldsymbol{w}$, for all $\boldsymbol{u} \in U$, and $\boldsymbol{w} \in \mathbb{R}^2 \cong T_{\boldsymbol{u}}(U)$, where

$$A = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

  with

$$E = \langle \phi_u, \phi_u \rangle\,, \quad F = \langle \phi_u, \phi_v \rangle\,, \quad G\,\langle \phi_v, \phi_v \rangle$$

  and

$$e = -\langle \phi_u, N_u \rangle\,, \quad f = -\langle \phi_u, N_v \rangle = -\langle \phi_v, N_u \rangle\,, \quad g = -\langle \phi_v, N_v \rangle$$

- In local coordinates the second fundamental form $\mathbf{II}_p = \mathbf{II}_{\phi(\boldsymbol{u})}$, is given by

$$\mathbf{II}_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{w}) = \langle \mathsf{D}\phi(\boldsymbol{u})\boldsymbol{v}, S_p(\mathsf{D}\phi(\boldsymbol{u})\boldsymbol{w}) \rangle = \left\langle \boldsymbol{v}, \begin{bmatrix} e & f \\ f & g \end{bmatrix} \cdot \boldsymbol{w} \right\rangle$$

  for all $\boldsymbol{u} \in U$, and $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2 \cong T_{\boldsymbol{u}}(U)$

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Transformation Properties of the Second Fundamental Form

Let σ be a parametrised surface.

- The second fundamental form is invariant under Euclidean transformations of $\mathbb{R}^3$: For $\psi : x \to Ax + b, A \in O(3, \mathbb{R}), b \in \mathbb{R}^3$, and $\tilde{\sigma} := \psi \circ \sigma$ one has

$$\tilde{\mathbf{II}}_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{w}) = \mathbf{II}_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{w}) \cdot \mathsf{sgn} \det \mathsf{D}\psi$$

  where $\mathbf{II}_{\boldsymbol{u}}$ is second fundamental form of $\tilde{\sigma}$ and $\mathsf{sgn} \det \mathsf{D}\psi = \mathsf{sgn} \det A$

- The second fundamental form transforms under reparametrisations as follows: Let $\tilde{\sigma} := \sigma \circ \phi$, $\phi : \tilde{D} \to D$, then

$$\tilde{\mathbf{II}}_{\boldsymbol{u}}(\boldsymbol{v}, \boldsymbol{w}) = \mathbf{II}_{\phi(\boldsymbol{u})}(\mathsf{D}\phi(\boldsymbol{v}), \mathsf{D}\phi(\boldsymbol{w})) \cdot \mathsf{sgn} \det \mathsf{D}\phi$$

  where $\tilde{\mathbf{II}}_{\boldsymbol{u}}$ is second fundamental form of $\tilde{\sigma}$

Normal Curvature

Let M by an oriented regular surface having Gaussian map N. Let $p \in M$ and $\boldsymbol{v} \in T_pM$ s.t. $||\boldsymbol{v}|| = 1$.

- The normal curvature $\kappa_n(\boldsymbol{v})$ of M at p in the direction of $\boldsymbol{v}$ is defined by

$$\kappa_n(\boldsymbol{v}) = \langle \gamma''(0), N(p) \rangle \,,$$

  with γ any regular curve in arc-length parametrisation s.t. $\gamma(0) = p, \gamma'(0) = \boldsymbol{v}$

- Recall that if $\gamma : I \to M$ be a curve parametrised by arc-length with $\gamma(0) = p$ and If $\gamma''(0) = \gamma''_\theta(0) + \gamma''_\nu(0)$ with $\gamma''_\theta(0) \in T_pM$ and $\gamma''_\nu(0) \perp T_pM$, then

$$\gamma''_\nu(0) = \langle \gamma''(0), N(p) \rangle \, N(p) = - \langle \gamma'(0), DN(p)\gamma'(0) \rangle \, N(p)$$

- Therefore, the normal curvature $\kappa_n(\boldsymbol{v})$ of M at p in the direction of $\boldsymbol{v}$ satisfies

$$\kappa_n(\boldsymbol{v}) = \langle \boldsymbol{v}, S_p(\boldsymbol{v}) \rangle = \mathbf{II_p}(\boldsymbol{v}, \boldsymbol{v})$$

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Principal Curvatures

Let M by an oriented regular surface having Gaussian map N.

- With $T_p^1 M := \{\boldsymbol{v} \in T_p M : ||\boldsymbol{v}|| = 1\}$, $\kappa_n : T_p^1 M \to \mathbb{R}$ is a continuous map
- There exist two directions $\boldsymbol{v}_1, \boldsymbol{v}_2 \in T_p^1 M$ s.t.

$$\kappa_1(p) := \kappa_n(\boldsymbol{v}_1) = \max_{\boldsymbol{v} \in T_p^1 M} \kappa_n(\boldsymbol{v})$$

$$\kappa_2(p) := \kappa_n(\boldsymbol{v}_2) = \min_{\boldsymbol{v} \in T_p^1 M} \kappa_n(\boldsymbol{v})$$

called principal curvatures. $\boldsymbol{v}_1, \boldsymbol{v}_2$ are called principal directions

Principal Curvatures

Since S_p is symmetric (thus also $\mathbf{II}_p$), it follows from the spectral theorem:

- There exists an orthonormal basis $\boldsymbol{v}_1, \boldsymbol{v}_2$ of T_pM s.t.

$$S_p(\boldsymbol{v}_1) = \lambda_1 \boldsymbol{v}_1 \quad S_p(\boldsymbol{v}_2) = \lambda_2 \boldsymbol{v}_2$$

  for $\lambda_1, \lambda_2 \in \mathbb{R}$

- $\boldsymbol{v} \in T_p^1 M$ is a principle direction if and only if it is an eigenvector of the shape operator S_p (thus $\lambda_1 = \kappa_1, \lambda_2 = \kappa_2$).

Gaussian and Mean Curvature

Let M by an oriented regular surface having Gaussian map $N : M \to \mathbb{S}^2$, and $S_p : T_pM \to T_pM$ be the shape operator.

- Gaussian curvature:
$$\mathbf{K}(p) = \det S_p$$
- Mean Curvature:
$$\mathbf{H}(p) = \frac{1}{2}\,\text{trace}\,(S_p)$$
- Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be an orthonormal basis of T_pM s.t.
$$S_p(\boldsymbol{v}_1) = \lambda_1 \boldsymbol{v}_1 \quad \text{and} \quad S_p(\boldsymbol{v}_2) = \lambda_2 \boldsymbol{v}_2,$$
then λ_1, λ_2 are principal curvatures and
$$\mathbf{K}(p) = \lambda_1 \lambda_2 = \kappa_1 \kappa_2, \quad \mathbf{H} = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{1}{2}(\lambda_1 + \lambda_2).$$
- The surface M is said to be flat if $\mathbf{K}(p) = 0$ for all $p \in M$ and minimal if $\mathbf{H}(p) = 0$ for all $p \in M$.

Gaussian and Mean Curvature

Let $\alpha_1, \alpha_2 : I \to M$ be curves s.t. $\alpha_1'(0) = \boldsymbol{v}_1$ and $\alpha_2'(0) = \boldsymbol{v}_2$ and

$$\kappa_1 = \langle S_p(\boldsymbol{v}_1), \boldsymbol{v}_1 \rangle =< \alpha_1''(0), N(p) >$$
$$\kappa_2 = \langle S_p(\boldsymbol{v}_2), \boldsymbol{v}_2 \rangle =< \alpha_2''(0), N(p) >$$

- If $\mathbf{K}(p) = \kappa_1 \kappa_2 > 0$ the curves α_1, α_2 stay locally on the same side of the tangent plane. Thus, all curves going through p stay locally on the same side of the plane
- If $\mathbf{K}(p) = \kappa_1 \kappa_2 < 0$ they stay locally on different sides of the tangent plane

We call p an umbilic point if $\kappa_1(p) = \kappa_2(p)$.

- If every point of a path-connected oriented regular surface M is an umbilic point, then M is either contained in a plane or in a sphere.

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Local Parametrisation

- Let $\phi : U \to \phi(U)$ be a local parametrisation of an oriented regular surface M.
- In local coordinates the gaussian and mean curvatures are given by

$$\mathbf{K}(p) = \frac{eg - f^2}{EG - F^2}$$

$$\mathbf{H}(p) = \frac{1}{2} \cdot \frac{eG - efF + gE}{EG - F^2}$$

with e, f, g, E, F, G defined in slide 4

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Curvature of Surfaces

- Definition of principal curvatures relies on the embedding of the surface into $\mathbb{R}^3$
- Principal curvatures therefore depend on this embedding, and change under isometric deformations of the surface
- Is there a quantity that depends only on the Riemannian manifold structure of the surface, i.e., it is independent on the embedding?

Gauss' Theorem

The Gaussian curvature $\mathbf{K}$ of a surface depends only on its inner metric.

Consequences:

- The Gaussian curvature of a 2D manifold embedded into $\mathbb{R}^3$ does not depend on the embedding
- Isometric deformations of a surface in $\mathbb{R}^3$ do not change the Gaussian curvature
- The metric of a surface (given by its first fundamental form) is (pointwise) hyperbolic, planar or elliptic

Outlook

- Surface evolutions
- Curvature motion processes for surfaces
- Diffusion on surfaces
- Diffusion smoothing of surfaces

References

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- W. Haack: Differential-Geometrie, Teil I. Wolfenbtteler Verlagsanstalt, Wolfenbttel 1948 (in German)