Lecture 8

- Regular Surfaces as 2-Dimensional Manifolds
- Constructing Surfaces (Implicit Mapping Theorem)
- Shape Operator
- Second Fundamental Form

Topological concepts (Continuous Mappings)

- Let $U \subset \mathbb{R}^2$ be an open set, and $\phi : U \to \mathbb{R}^3$.
- Assume that $\phi : U \to \phi(U)$ is bijective, with inverse $\psi = \phi^{-1}$. When is ψ continuous?
    - Consider the induced topology of the Euclidean space $\mathbb{R}^3$ on $\phi(u)$:
      A subset of $S \subset M$ is open if $S = M \cap V$ for some open set $V \subset \mathbb{R}^3$.
    - ψ is continuous if $\psi^{-1}(A)$ is an open set in the topology induced on M for every open set $A \subset U$
- If both ϕ and ψ are continuous they are called homeomorphisms

Regular Surfaces

- $M \subset \mathbb{R}^3$ is a 2-dimensional manifold (regular surface), if for each point $p \in M$ there exist open, connected and simply connected neighbourhoods $U \subset \mathbb{R}^2$, $V \subset \mathbb{R}^3$ with $p \in V$ and a bijective smooth map $\phi : U \to V \cap M$ such that ϕ is a homeomorphism and

$$\phi_u(p) \times \phi_v(p) \neq 0$$

  for all $p \in U$.

- We call any such ϕ a local parametrisation. $\phi^{-1} : \phi(U) \to U$ is a chart
- A collection $\mathcal{A} := \left\{(V_\alpha \cap M, \phi_\alpha^{-1}), \alpha \in I\right\}$ is an atlas of M if

$$M = \bigcup_\alpha (V_\alpha \cap M)$$

Remarks:

- To distinguish we will call *regular surfaces* the 2-dim manifold embedded in $\mathbb{R}^3$, and *parametrised surfaces* the regular maps $\sigma : D \to \mathbb{R}^2$ of the previous lecture.
- Here and in what follows smooth mean sufficiently differentiable. If less regularity is required it will specified

Inverse Mapping Theorem

- Let $r > 0$, $U \subset \mathbb{R}^n$, and $F : U \to \mathbb{R}^n$ be C^r with $\mathsf{D}F(p) : \mathbb{R}^n \to \mathbb{R}^n$ invertible.
- Then F is locally invertible: $F|U_p : U_p \to U_q$ is a bijection with C^r inverse $f : U_q \to U_p$ in open neighborhoods U_p of U_q of p and $q = F(p)$
- $\mathsf{D}(f^{-1}(q)) = (\mathsf{D}f(p))^{-1}$

Implicit Mapping Theorem

- Let $f : U \to \mathbb{R}$ be C^1, for U open subset of $\mathbb{R}^3$.
- Let q be a regular value of f, i.e. $(\nabla f)(p) \neq 0$ for all p in $M = f^{-1}(q)$.
- Then M is a regular surface in $\mathbb{R}^3$.

Examples:

- (Sphere) $F : \mathbb{R}^3 \to \mathbb{R}$, $F(x, y, z) = x^2 + y^2 + z^2$
- (Torus) $F : \{(x, y, z) : x^2 + y^2 \neq 0\} \to \mathbb{R}$, $F(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - R)^2$

Differentiable map

$F : M_1 \to M_2$ is differentiable at p if there exist local parametrisations $\phi_p : U_p \to \phi_p(U_p)$ and $\phi_q : U_q \to \phi(U_q)$, around p and $q = F(p)$ s.t.

$$\phi_q^{-1} \circ F \circ \phi_p \quad \text{is differentiable.}$$

- This definition is independent on the choice of local parametrisations
- The composition of differentiable maps between surfaces is again differentiable
- If F is bijective with differentiable inverse it is called diffeomorphism
- Local parametrisations $\phi : U \to \phi(U)$ are diffeomorphisms
- M_1, M_2 regular surfaces and $\phi : U \to \mathbb{R}^3$ differentiable with $U \in \mathbb{R}^3$ open s.t. $M_1 \subset U, M_2 \subset \phi(U)$. Then $\phi|_{M_1} : M_1 \to M_2$ is differentiable

Tangent Space of a Surface

- Let M be a regular surface and $p \in M$. The tangent space T_pM of M at p is the set of all tangents $\gamma'(0)$ to differentiable curves $\gamma : I \to M$ such that $\gamma(0) = p$
- The tangent space T_pM of a regular surface is a 2-dimensional real vector space.
- Let $\phi : M_1 \to M_2$ be differentiable between regular surfaces with $p \in M_1$ and $q \in M_2$, and $\phi(p) = q$. Then

$$\mathrm{D}\phi(p) : \gamma'(t) \to \frac{d}{dt}(\phi \circ \gamma)|_{t=0}$$

determines a well defined linear map $\mathrm{D}\phi(p) : T_pM_1 \to T_qM_2$, the differential or tangent map of ϕ at p.

First Fundamental Form of a Regular Surface

Let M, M_1, M_2 be regular surfaces.

- First fundamental form $I_p : T_pM \times T_pM \to \mathbb{R}$ is given by the restriction of the Euclidean scalar product of $\mathbb{R}^3$ to T_pM.
- For any local parametrisation around $p \in M$, $\phi : U \to \phi(M)$, the differential $\mathrm{D}\phi(p)$ "pulls back" the Euclidean scalar product to $T_{\phi^{-1}(p)}U$
- In the local parameter region U the first fundamental form leads to a metric induced by the scalar product

$$< A(q)\cdot, \cdot >: T_{\phi^{-1}(p)}U \times T_{\phi^{-1}(p)}U \cong \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

where

$$A(q)_{11} =< \phi_u(q), \phi_u(q) >, \qquad A(q)_{22} =< \phi_v(q), \phi_v(q) >,$$
$$A(q)_{12} = A(q)_{21} =< \phi_u(q), \phi_v(q) >$$

for all $q = \phi^{-1}(p)$

Isometric maps

A differentiable $\phi : M_1 \to M_2$ is isometric if $\mathsf{D}\phi(p) : T_p M_1 \to T_{\phi(p)} M_2$ preserves the first fundamental form:

$$\tilde{\mathbf{I}}_{\phi(\mathbf{p})}(\mathsf{D}\phi(\mathbf{p})v, \mathsf{D}\phi(\mathbf{p})w) = \mathbf{I}_{\mathbf{p}}(\mathbf{v}, \mathbf{w})$$

- Euclidean transformations are isometric
- Local parametrisation is isometric

Gaussian Map

- Let M be a regular surface. A differentiable map $N : M \to \mathbb{S}^2$ is said to be a Gauss map for M if $N(p) \perp T_pM$, for each $p \in M$
- M is said to be orientable if such a Gauss map exists. A surface M equipped with a Gauss map is said to be oriented
- Equivalently, M is orientable if it has an atlas with only changes of charts which are orientation-preserving $(\det > 0)$
- Let $\gamma : I \to M$ be a curve parametrised by arc-length with $\gamma(0) = p$. If $\gamma''(0) = \gamma''_\theta(0) + \gamma''_\nu(0)$ with $\gamma''_\theta(0) \in T_pM$ and $\gamma''_\nu(0) \perp T_pM$. Then

$$\gamma''_\nu(0) = -\langle \gamma'(0), DN(p)\gamma'(0) \rangle N(p)$$

Shape Operator

Let M by an oriented regular surface having Gaussian map N.

- we can identify $T_pM \cong T_{N(p)}\mathbb{S}^2$
- The shape operator $S_p : T_pM \to T_pM$ is the linear map given by $S_p(\boldsymbol{v}) = -\mathsf{D}N(p)(v)$, for all $\boldsymbol{v} \in T_pM$.
- The shape operator is symmetric:

$$\langle S_p(\boldsymbol{v}), \boldsymbol{w} \rangle = \langle \boldsymbol{v}, S_p(\boldsymbol{w}) \rangle \,, \quad \boldsymbol{v}, \boldsymbol{w} \in T_pM$$

Second Fundamental Form

Let M by an oriented regular surface having Gaussian map N.

- The second fundamental form of M at p, $S_p : T_pM \times T_pM \to \mathbb{R}$ is given by $\mathbf{II_p}(\boldsymbol{v}, \boldsymbol{w}) = \langle S_p(\boldsymbol{v}), \boldsymbol{w} \rangle \,, \quad \boldsymbol{v}, \boldsymbol{w} \in T_pM$
- $\mathbf{II_p}$ is a symmetric bilinear map