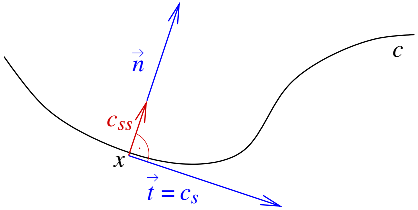
Curves and Surfaces in Euclidean Space

- Curves in $\mathbb{R}^3$ and $\mathbb{R}^d$, Frenet frames
- Surfaces in $\mathbb{R}^3$, Gauss frames
- First and second fundamental form

Reminder about Curves and Curvature in 2D

- In 2D, a regular curve is characterised up to Euclidean transformations by the curvature κ (as function of the curve parameter)
- At each curve point, there are tangent and normal unit vectors $\vec{t}$ and $\vec{n}$, such that in arc-length parametrisation

$$c_s = \vec{t}\,, \qquad c_{ss} = \kappa \vec{n}$$



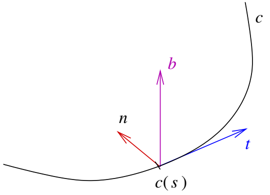
Curve with tangent and normal vectors and first two derivatives at a point, from Lecture 2

Curvature Parameters of a Curve in 3D

- Consider regular curve in 3D Euclidean space, $c : I \to \mathbb{R}^3$, in arc-length parametrisation
- At each curve point $c(s)$, there are
  - a unit tangent vector $\vec{t}(s)$, $\quad c_s = \vec{t}(s)$
  - a unit normal vector $\vec{n}(s)$, $\quad c_{ss} = \kappa(s)\vec{n}(s)$
  - a unit **binormal** vector $\vec{b}(s)$, $\quad \vec{b}(s) = \vec{t}(s) \times \vec{n}(s)$

  (unique if c is 2-regular, i.e. $c_{ss} \neq 0$)

- In contrast to the 2D case, $\kappa(s)$ can always be chosen nonnegative
- $\vec{t}(s)$, $\vec{n}(s)$, $\vec{b}(s)$ form an orthonormal system, the **Frenet frame** of c



Curve c in $\mathbb{R}^3$ with Frenet frame at $c(s)$

Curvature Parameters of a Curve in 3D, cont.

- the derivative of $\vec{b}$ is perpendicular to both $\vec{t}$ and $\vec{b}$, i.e

$$\vec{b}_s(s) = -\tau(s)\vec{n}(s)$$

  with a function $\tau(s)$, the **torsion** of c

- Curvature $\kappa(s)$ and torsion $\tau(s)$ determine the curve $c(s)$ up to rotations and translations

- **Frenet-Serret equations** (or Frenet equations):

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}$$

- The torsion can also be defined by

$$c_{sss}(s) = -\kappa(s)^2\,\vec{t}(s) + \kappa_s(s)\,\vec{n}(s) + \kappa(s)\tau(s)\,\vec{b}(s)$$

- The torsion vanishes identically if and only if the curve is contained in a plane

Frenet Frames in Higher Dimensions

- Consider a curve c in $\mathbb{R}^d$ in arc-length parametrisation
- For each curve point $c(s)$, there is an orthonormal basis $(\vec{e}_1, \ldots, \vec{e}_d)$ such that

$$c_s(s) = \vec{e}_1 \;, \qquad c_{ss}(s) \in \mathrm{Span}\,(\vec{e}_1, \vec{e}_2) \;,$$

$$c_{s^{(k)}}(s) \in \mathrm{Span}\,(\vec{e}_1, \ldots, \vec{e}_k) \;, \quad k \le d$$

- $(\vec{e}_1, \ldots, \vec{e}_d)$ is the *Frenet frame* of c.
- *Frenet equations* in d dimensions:

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \\ \vdots \\ \\ \vec{e}_d \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \ldots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & & 0 & 0 \\ 0 & -\kappa_2 & 0 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & \kappa_{d-1} \\ 0 & 0 & 0 & \ldots & -\kappa_{d-1} & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \\ \vdots \\ \\ \vec{e}_d \end{pmatrix}$$

- *Curvature functions* $\kappa_i(s)$, $i = 1, \ldots, d-1$ (nonnegative for $i \le d-2$) determine c up to rotations and translations

Surfaces

- **Surface in $\mathbb{R}^d$:** Differentiable function $\sigma : D \to \mathbb{R}^d$, $D \subset \mathbb{R}^2$ connected domain
- **Graph (image) of a surface σ:** Set of points in $\mathbb{R}^d$ given by $\{\sigma(u,v) \mid (u,v) \in D\}$

Remark: Similarly as for curves, surfaces with identical graphs but different parametrisations are considered different.

Curves on a Surface

- $\sigma : D \to \mathbb{R}^3$ surface
- $I \subset \mathbb{R}$ interval, mapping $I \ni p \mapsto (u(p), v(p)) \in D$
- By $c : I \to \sigma(D) \subset \mathbb{R}^3, p \mapsto \sigma(u(p), v(p))$, a curve on σ is given

Related Definitions

- **Regular surface:** Surface $\sigma : (u, v) \mapsto \mathbb{R}^d$ is regular if the (Jacobi) matrix

$$\mathrm{D}\sigma := \begin{pmatrix} \sigma_u^1 & \sigma_v^1 \\ \vdots & \vdots \\ \sigma_u^d & \sigma_v^d \end{pmatrix}$$

  has rank 2 everywhere in D.

- k-regularity can be defined analogously using higher order derivatives. We will always assume that surfaces are sufficiently many times differentiable.

- **Tangent plane** $\mathrm{T}_{(u,v)}\sigma$ to σ at $\sigma(u, v)$: image of

$$\mathrm{D}\sigma(u, v) : \mathrm{T}_{(u,v)}\mathbb{R}^2 \to \mathrm{T}_{\sigma(u,v)}\mathbb{R}^d$$

Reparametrisation

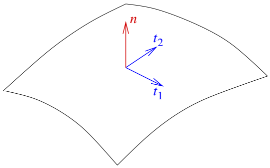
- Transforms a surface into another one with the same graph
- $\sigma : D \to \mathbb{R}^d$ surface
- $\tilde{D} \subset \mathbb{R}^2$ connected domain
- $\varphi : \tilde{D} \to D$ differentiable mapping with $\operatorname{rank} \mathrm{D}\varphi = 2$ everywhere
- $\tilde{\sigma} := \sigma \circ \varphi : \tilde{D} \to \mathbb{R}^d$ reparametrised surface
- **orientation-preserving** if $\det \mathrm{D}\varphi > 0$

Gauss Frame for Surfaces in 3D

- Restrict now to surfaces $\sigma : D \to \mathbb{R}^3$ in 3D Euclidean space
- **Gauss frame** (analog of Frenet frame for surfaces) consists of the three unit vectors

$$\vec{t}_1(u) := \frac{\sigma_u}{\|\sigma_u\|}\,, \qquad \vec{t}_2(u) := \frac{\sigma_v}{\|\sigma_v\|}\,, \qquad \vec{n} := \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

- First two vectors $\vec{t}_1$ and $\vec{t}_2$ of the frame lie in tangential direction, $\vec{n}$ perpendicular to the surface
- **In general, $\vec{t}_1$ and $\vec{t}_2$ are not orthogonal**
- $\vec{t}_1$ and $\vec{t}_2$ depend on parametrisation
- **Normal vector** $\vec{n}$ does not change under orientation-preserving reparametrisation, is reverted by orientation-changing reparametrisation



First Fundamental Form

- Consider regular surface $\sigma : D \to \mathbb{R}^3$
- Use boldface letters $\mathbf{u}, \mathbf{v}, \ldots$ for points and vectors in $\mathbb{R}^2$
- Symmetric bilinear form

$$\mathbf{I_u}(\mathbf{v}, \mathbf{w}) := \langle \mathrm{D}\sigma(\mathbf{u})\, \mathbf{v}, \mathrm{D}\sigma(\mathbf{u})\, \mathbf{w} \rangle \;, \quad \mathbf{v}, \mathbf{w} \in \mathrm{T}_u D$$

  is called **first fundamental form** of σ at $\mathbf{u}$

- Regularity implies $\mathbf{I_u}(\mathbf{w}, \mathbf{w}) \neq 0$, for nonzero $\mathbf{w}$
- In coordinates, $\mathbf{I_u}$ is described by a matrix which we will also denote by $\mathbf{I_u}$ $(\mathbf{u} = (u, v))$:

$$\mathbf{I_u} = \mathbf{I}_{(u,v)} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \;,$$

$$E = \langle \sigma_u, \sigma_u \rangle \;, \qquad F = \langle \sigma_u, \sigma_v \rangle \;, \qquad G = \langle \sigma_v, \sigma_v \rangle$$

- Map $\mathrm{D}\sigma(\mathbf{u}) : \mathrm{T}_{\mathbf{u}}\mathbb{R}^2 \to \mathrm{T}_{\mathbf{u}}\sigma \subset \mathrm{T}_{\sigma(\mathbf{u})}\mathbb{R}^3$ allows to transfer $\mathbf{I_u}$ also into a bilinear form on $\mathrm{T}_{\mathbf{u}}\sigma$

Transformation Properties of the First Fundamental Form

- The first fundamental form is **invariant under Euclidean transformations** (including reflections) of $\mathbb{R}^3$:
  For $\psi : x \mapsto Ax + b$, $A \in \mathrm{O}(3, \mathbb{R})$, $b \in \mathbb{R}^3$, and $\tilde{\sigma} := \psi \circ \sigma$ one has

$$\tilde{\mathbf{I}}_{\mathbf{u}}(\mathbf{v}, \mathbf{w}) = \mathbf{I}_{\mathbf{u}}(\mathbf{v}, \mathbf{w})$$

  where $\tilde{\mathbf{I}}_{\mathbf{u}}$ is first fundamental form of $\tilde{\sigma}$

- The first fundamental form **transforms under reparametrisations** as follows:
  Let $\tilde{\sigma} := \sigma \circ \varphi$, $\varphi : \tilde{D} \to D$; then

$$\tilde{\mathbf{I}}_{\mathbf{u}}(\mathbf{v}, \mathbf{w}) = \mathbf{I}_{\varphi(\mathbf{u})}(\mathrm{D}\varphi(\mathbf{v}), \mathrm{D}\varphi(\mathbf{w}))$$

  where $\tilde{\mathbf{I}}_{\mathbf{u}}$ is first fundamental form of $\tilde{\sigma}$

Measurements Using the First Fundamental Form

- Consider surface $\sigma : D \to \mathbb{R}^3$ and a curve $c : [0, P] \to D$ on σ. Then

$$\mathbf{I}_{(u(p),v(p))}(c_p(p), c_p(p)) = c_p^{\mathrm{T}} \mathbf{I}_{(u(p),v(p))} c_p = E\, u_p^2 + 2F\, u_p v_p + G\, v_p^2$$

  Length of c:

$$L[c] = \int_0^P \|c_p(p)\| \; \mathrm{d}p = \int_0^P \sqrt{E\, u_p^2 + 2F\, u_p v_p + G\, v_p^2} \; \mathrm{d}p$$

- **Angle** between vectors in a point (here for σ_u, σ_v)

$$\cos \measuredangle(\sigma_u, \sigma_v) = \frac{\langle \sigma_u, \sigma_v \rangle}{\|\sigma_u\| \; \|\sigma_v\|} = \frac{F}{\sqrt{EG}}$$

Measurements Using the First Fundamental Form, cont.

- **Area** of surface $\sigma(D)$:

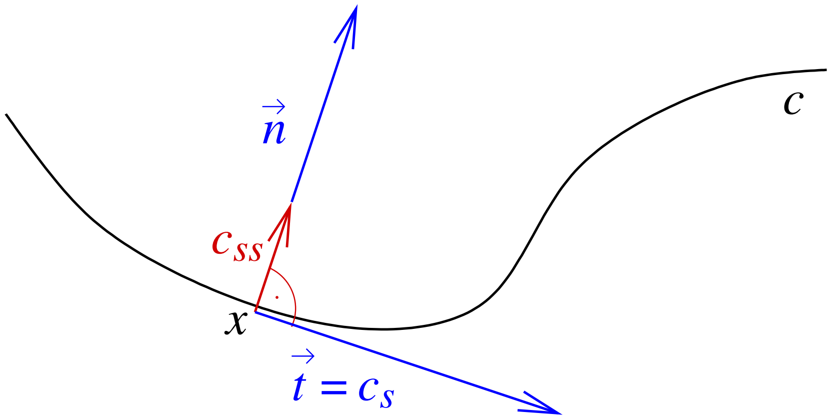
$$A[\sigma] = \iint_D \sqrt{\det \mathbf{I}_{(u,v)}}\, \mathrm{d}u\, \mathrm{d}v = \iint_D \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v$$

- **The bilinear form on $\mathrm{T}_{(u,v)}\sigma$ defined by the first fundamental form is a Riemannian metric on the surface (graph). It is obtained by restricting the Euclidean metric of $\mathbb{R}^3$ to the surface.**

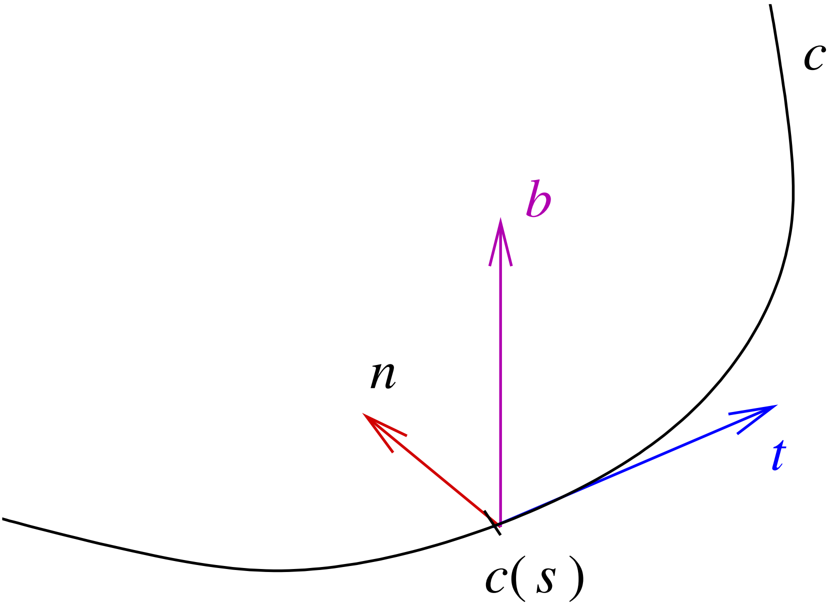
Outlook and References

References

- G. Sapiro: *Geometric Partial Differential Equations and Image Analysis.* Cambridge University Press 2001
- W. Haack: *Differential-Geometrie, Teil I.* Wolfenbütteler Verlagsanstalt, Wolfenbüttel 1948 (in German)



$\vec{n}$
c
c_{ss}
x
$\vec{t} = c_s$


c
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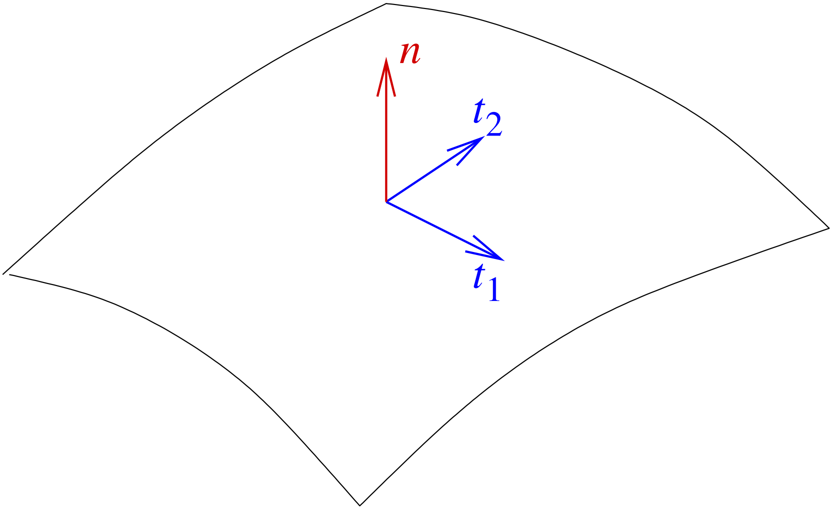
n
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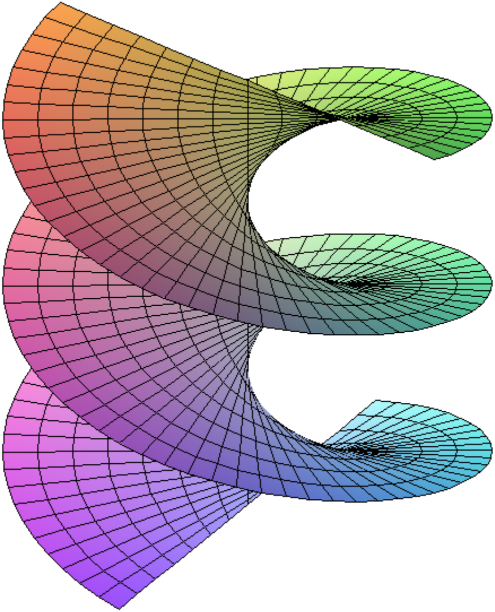

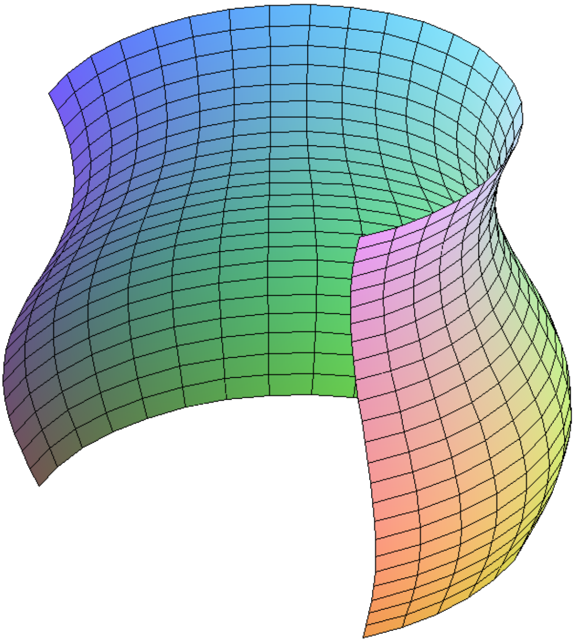

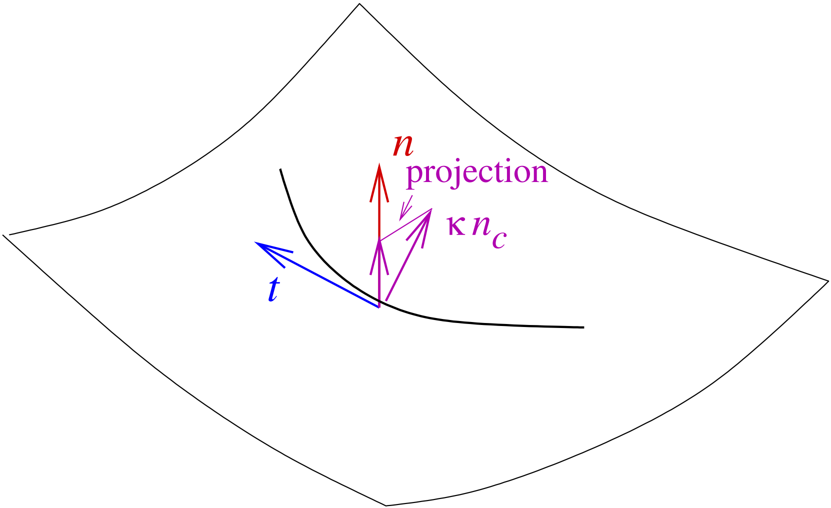
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projection
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