Lecture 6

- Transformations, Invariances, and Lie Groups
- Affine Differential Geometry
- Affine Curvature Motion
- Some Sobolev Norms
- Sobolev Gradient Descents for Curves

What did we do last week?

- energy functionals

$$E(u) := \int_D F(x, u, u_{x_i})\,\mathrm{d}x$$

- variational gradient $\frac{\delta}{\delta_u} E(u) := g$ with

$$\begin{aligned}\delta_v E &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} E(u^* + \varepsilon v)\Big|_{\varepsilon=0} \\ &= \langle v, g\rangle\end{aligned}$$

- gradient descent

$$\partial_t u = -g = -\frac{\delta}{\delta u} E$$

Motivation

- Lengths, angles and curvatures were the most important quantities in our geometric considerations up to now.

  What is special about lengths, angles and curvatures?

- **Intuition:** These quantities describe object properties which do not change under some geometric transformations (and are therefore considered more characteristic for the objects).

- **Mathematical notion:** These quantities are **invariant** under translations, rotations (and in some cases also re-scalings).

- **Idea:** The essential geometric structure of a space (manifold, image domain) is given by its "admissible" transformations.

Transformation Groups

- Reasonable requirements for admissible transformations:
  - Concatenation of admissible transformations is again an admissible transformation, and follows the associative law
  - There is a trivial transformation which changes nothing
  - For each admissible transformation, there is an admissible transformation that reverts its effect

$\Rightarrow$ Transformations must form a **group.**

Euclidean Transformation Group

- Matrix group $\mathrm{SO}(d, \mathbb{R})$:
  - Consists of all real $d \times d$ matrices A with $AA^{\mathrm{T}} = A^{\mathrm{T}} A = I$ (unit matrix) and $\det A = 1$
  - Describes the (true, i.e. orientation-preserving) rotations around the origin in $\mathbb{R}^d$
- **Euclidean transformation group:** Formed by translations, rotations and re-scalings

$$\tilde{x} = kAx + b \,, \qquad A \in \mathrm{SO}(d, \mathbb{R}) \,, \quad b \in \mathbb{R}^d \,, \quad k \in \mathbb{R}$$

- The differential geometry built on notions invariant under these transformations is called **Euclidean differential geometry**

Lie Groups

- As seen before, transformations of a geometric space need to form a group
- Typically, transformations also form a differentiable manifold, and group operations (concatenation=group multiplication, inversion) are differentiable

  Example: Euclidean transformations of $\mathbb{R}^d$,

  $$\tilde{x} = kAx + b \,, \qquad A \in \mathrm{SO}(d,\mathbb{R}) \,, \quad b \in \mathbb{R}^d \,, \quad k \in \mathbb{R} \,,$$

  are parametrised by $\frac{1}{2}(d^2+d+2)$ real parameters and form therefore a $\frac{1}{2}(d^2+d+2)$-dimensional differentiable manifold
- **Lie group:**
  - Differentiable manifold L
  - L is also a group
  - Group operations of L are differentiable

Sophus Lie

Sophus Lie (1842–1899), Norwegian mathematician. Studied the transformation groups under which differential equations and their solutions are invariant, later named Lie groups. *(Image: public domain, source: Wikipedia)*

Can a "reasonable" differential geometry be developed with another Lie group in place of the Euclidean transformation group?

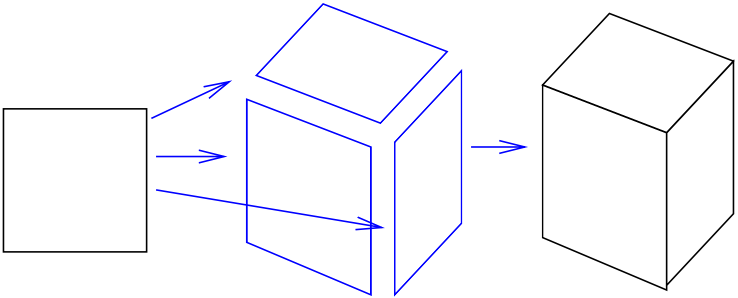
The Affine Transformation Group

- **Matrix group $\mathrm{GL}^+(d, \mathbb{R})$:**
  - Consists of all real $d \times d$ matrices A with $\det A > 0$
  - Describes all non-degenerate orientation-preserving transformations of $\mathbb{R}^d$ which leave the origin fixed
- **Affine transformation group:** Formed by all affine transformations of $\mathbb{R}^d$ (i.e. translations, rotations, re-scalings, shears),

$$\tilde{x} = Ax + b \,, \qquad A \in \mathrm{GL}^+(d, \mathbb{R}) \,, \quad b \in \mathbb{R}^d$$

- Differential geometry based on concepts invariant under this group is called **affine differential geometry**

- **Relation to Image Processing:** Unlike the Euclidean transformation group, the affine one can cope with the appearance changes of planar shapes when they are seen from different (parallel) perspectives.



Planar Affine Curve Theory

- Let a curve $c : I \to \mathbb{R}^2$, $p \mapsto (x(p), y(p))$ be given, I interval. Assume c is strictly convex
- Euclidean geometry: Arc-length parametrisation gives a distinguished description of the curve
- Affine geometry: Which parametrisation is advantageous?

Affine Metric and Arc Length

- Consider $g(p)$ defined by

$$(g(p))^3 := \det(c_p, c_{pp}) = \det \begin{pmatrix} x_p & x_{pp} \\ y_p & y_{pp} \end{pmatrix}$$

- $g(p)$ is invariant under *special affine transformations*

$$\tilde{x} = Ax + b\ , \qquad\qquad \det A = 1$$

- Choose therefore

$$c(p) = \tilde{c}(s)\ , \qquad\qquad \det(\tilde{c}_s, \tilde{c}_{ss}) = 1$$

- New parameter

$$s(p) = \int_0^p (g(\tau))\,\mathrm{d}\tau = \int_0^p (\det(c_p(\tau), c_{pp}(\tau)))^{1/3}\,\mathrm{d}\tau$$

- s is called **affine invariant arc length**
- g is called **affine metric**

Conversion Formulas

- Conversion between general parametrisation and affine arc-length parametrisation

curve parameter	$\mathrm{d}s = g\,\mathrm{d}p\;,$
affine tangential vector	$\vec{t} = c_s = c_p \dfrac{\mathrm{d}p}{\mathrm{d}s} = g^{-1} c_p\;,$
affine normal vector	$\vec{n} = c_{ss} = c_{pp}\left(\dfrac{\mathrm{d}p}{\mathrm{d}s}\right)^2 + c_p \dfrac{\mathrm{d}^2 p}{\mathrm{d}s^2} = g^{-2} c_{pp} - g^{-3} g' c_p$

- Conversion between Euclidean and affine arc-length parameters

$$\mathrm{d}s = \kappa^{1/3}\,\mathrm{d}s_{\mathrm{e}}\;,$$

$$\vec{t} = \kappa^{-1/3} \vec{t}_{\mathrm{e}}\;,$$

$$\vec{n} = \kappa^{1/3} \vec{n}_{\mathrm{e}} + f(\kappa, \kappa') \vec{t}_{\mathrm{e}}$$

where s_{e}, $\vec{t}_{\mathrm{e}}$, $\vec{n}_{\mathrm{e}}$ denote Euclidean arc-length parameter, tangential and normal vector, and f is some function of (Euclidean) curvature κ and its derivative along the curve κ'

Affine Curvature

- Consider curve c in affine arc-length parametrisation
- Differentiation of

$$\det(c_s, c_{ss}) = 1$$

w.r.t. s gives

$$\det(c_s, c_{sss}) = 0\ .$$

- Thus

$$c_{sss} + \mu c_s = 0$$

with

$$\mu = \det(c_{ss}, c_{sss}) = -\det(c_s, c_{ssss})$$

- μ is called **affine curvature**
- In general parametrisation

$$\mu = g^{-5}\Big(\det(c_{pp}, c_{ppp}) - g^{-1}g'\det(c_p, c_{ppp}) + g^{-1}g''\det(c_p, c_{pp})\Big)$$

Affine Invariance

- Under *special* affine transformations, $\mathrm{d}s$ and μ are absolutely invariant
- For *general* affine transformations $\tilde{x} = Ax + b$ we have *relative invariance*

$$
\begin{aligned}
\mathrm{d}\tilde{s} &= (\det A)^{1/3}\,\mathrm{d}s \\
\tilde{c}_{\tilde{s}} &= A(\det A)^{-1/3} c_s \\
\tilde{c}_{\tilde{s}\tilde{s}} &= A(\det A)^{-2/3} c_{ss} \\
\tilde{\mu} &= (\det A)^{-2/3} \mu
\end{aligned}
$$

Curves of Constant Affine Curvature

Curves of constant affine curvature: *conics*

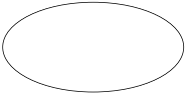
- Ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

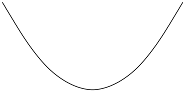
  affine arc-length parametrisation $x = a\cos\dfrac{s}{\sqrt[3]{ab}}$, $y = b\sin\dfrac{s}{\sqrt[3]{ab}}$

  affine curvature $\mu = \dfrac{1}{\sqrt[3]{a^2b^2}}$

- Parabola $y = \dfrac{1}{2}x^2$

  affine arc-length parametrisation $x = s$, $y = s^2/2$
  affine curvature $\mu = 0$





Affine Curvature Motion

- Consider evolution of *convex* curves c
- With affine arc-length parameter s,

$$c_t = c_{ss}$$

  describes **affine curvature motion**

- Rewrite using

$$\vec{n} = \kappa^{1/3} \vec{n}_{\mathrm{e}} + f(\kappa, \kappa') \vec{t}_{\mathrm{e}}$$

  where $\vec{n}$, $\vec{n}_{\mathrm{e}}$ affine/Euclidean normal vectors

- This gives

$$c_t = \kappa^{1/3} \vec{n}_{\mathrm{e}}$$

- In this form, affine curvature flow can be applied even to non-convex curves
- Note that affine curvature motion is here expressed in Euclidean terms, and that it is a curvature-dependent flow, thus morphologically invariant

Properties of Affine Curvature Motion

- Affine curvature motion is a gradient descent for affine arc-length
- Each simple closed curve evolves into a convex curve and vanishes as an
- For affine metric and curvature

$$g = \left(\det\left(c_p, c_{pp}\right)\right)^{1/3} , \qquad \mu = \det\left(c_{ss}, c_{sss}\right) ,$$

one has

$$g_t = -\frac{2}{3} g\mu , \qquad \mu_t = \frac{4}{3}\mu^2 + \frac{1}{3}\mu_{ss}$$

- If $\mu(p, 0) > 0$, then $\mu(p, t) > 0$ for all t
- Even curve segments with $\mu = 0$ at $t = 0$, enclosed between segments with $\mu > 0$, vanish instantaneously
- Euclidean curvature κ evolves under affine curvature motion according to $\kappa_t = \mu\kappa$
- Euclidean length l of a closed curve decreases, $l_t < 0$ if $l > 0$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34

Affine Curvature Motion – Special Case

- Consider ellipse in affine arc-length parametrisation

$$c_0 = \begin{pmatrix} a\cos(\sqrt{\mu}\, s) \\ b\sin(\sqrt{\mu}\, s) \end{pmatrix}, \qquad \mu = \mu_0 = (ab)^{-2/3}, \quad a, b > 0$$

- In each point, $\mu_s = \mu_{ss} = 0$ and

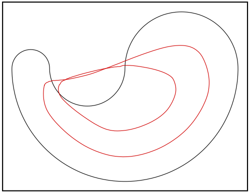
$$\mu_t = \frac{4}{3}\mu^2, \qquad \mu(t) = \frac{\mu_0}{1 - \frac{4}{3}\mu_0 t}$$

$$\kappa_t = \mu\kappa = \frac{\mu_0\kappa}{1 - \frac{4}{3}\mu_0 t}, \qquad \kappa(p,t) = \frac{\kappa(p,0)}{\left(1 - \frac{4}{3}\mu_0 t\right)^{3/4}}$$

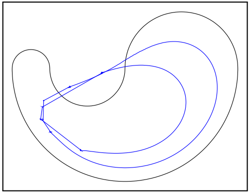
- Consequently, the initial ellipse with semi-axes a, b and affine curvature $\mu_0 = (ab)^{-2/3}$ evolves into *similar* ellipses
- Similarity factor at time t:

$$\alpha(t) = \left(1 - \frac{4}{3}\mu_0 t\right)^{3/4}$$

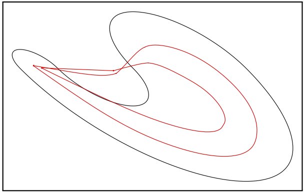
Example: Comparison to Euclidean Curvature Motion



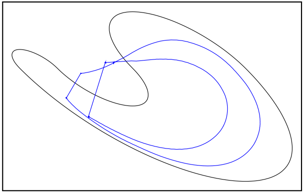
Original curve and two progressive stages of affine curvature motion.



Euclidean curvature motion. Straight line segments are artifacts because no resampling was implemented.

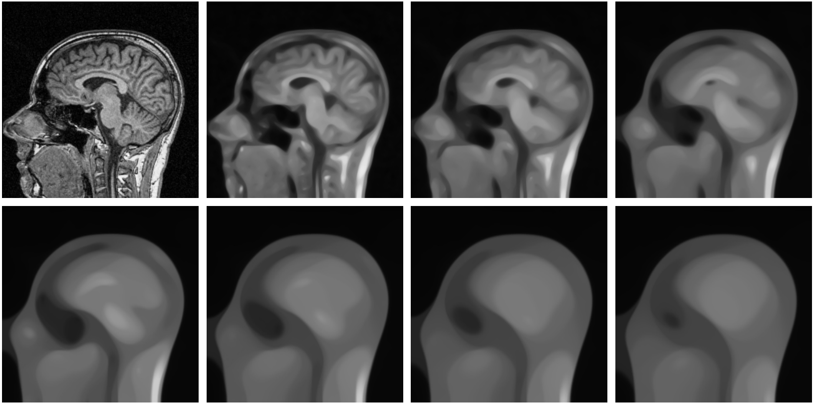


Affine curvature motion of a sheared version of the original curve. Evolved curves are (approx.) sheared versions of the evolved curves in the top example.



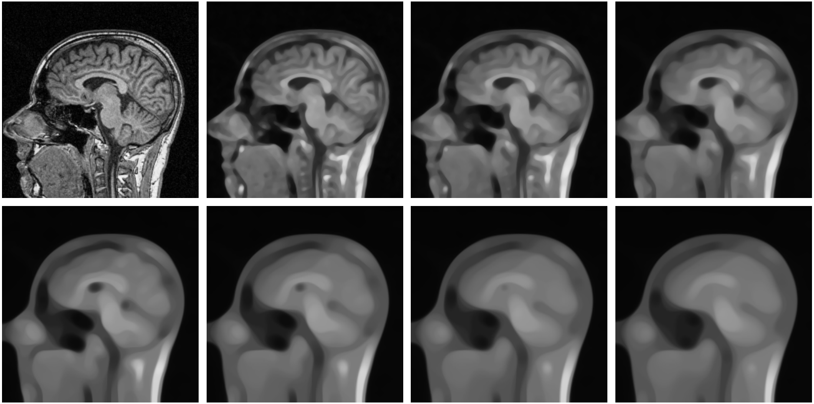
Euclidean curvature motion of the sheared curve. No shear invariance is observed.

Image Evolution with Affine Curvature Motion



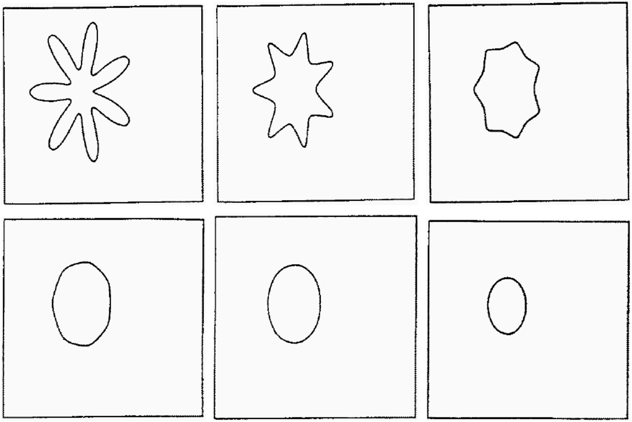
Top left to bottom right in rows: Original MR image and affine curvature motion at evolution times $t = 5$, $t = 10$, $t = 20$, $t = 40$, $t = 60$, $t = 80$, and $t = 100$.

Image Evolution Example with Euclidean Curvature Motion (for Comparison)



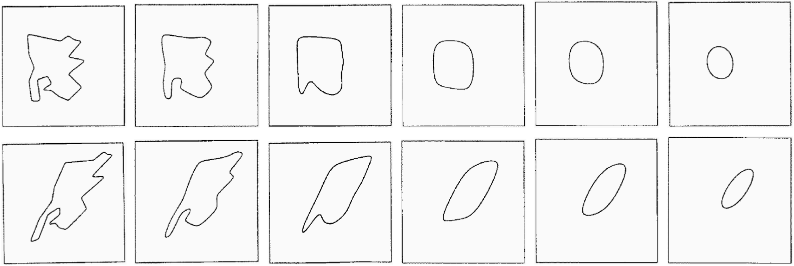
Top left to bottom right in rows: Original MR image and Euclidean curvature motion at evolution times $t = 5$, $t = 10$, $t = 20$, $t = 40$, $t = 60$, $t = 80$, and $t = 100$. Compare also Slide 3:30

Further Examples



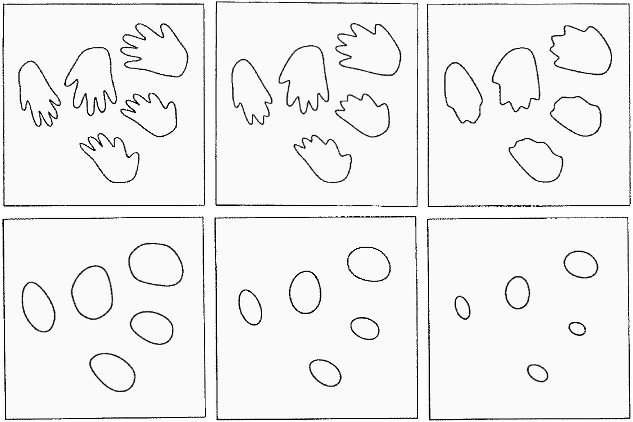
Left to right in rows: Evolution of a star-shaped initial contour under affine curvature flow. The shape becomes convex, shrinks and approaches an elliptic shape. *(Sapiro and Tannenbaum, 1993)*

Further Examples



Top, left to right: Evolution of an initial contour under affine curvature flow. **Bottom:** Same for an initial contour which results from the first by unimodular affine transformation. The affine relation between the initial contours is preserved for all later times. *(Sapiro and Tannenbaum, 1993)*

Further Examples



Left to right in rows: Evolution of five hand shapes related by affine transformations under affine curvature flow. Again the affine relation is preserved during evolution. *(Sapiro and Tannenbaum, 1993)*

Some Sobolev Spaces

Let $I = [0, L]$ with identification of endpoints (circle line). Consider functions $I \to \mathbb{R}^d$ which are sufficiently smooth.

- scaled "Standard" inner product: scaled L^2 product

$$\langle u, v \rangle_{H^0} = \frac{1}{L} \int_0^L u(x) \cdot v(x) \, \mathrm{d}x$$

  - Function space consists of such functions u for which $\langle u, u \rangle < \infty$
  - In this context, we call the resulting space $H^0(I)$

- Define

$$\langle u, v \rangle_{H^1} := \langle u, v \rangle_{H^0} + \lambda L^2 \langle u', v' \rangle_{H^0}$$
$$\langle u, v \rangle_{\tilde{H}^1} := \bar{u} \cdot \bar{v} + \lambda L^2 \langle u', v' \rangle_{H^0}$$

with $\bar{u} := \frac{1}{L} \int_0^L u(x) \, \mathrm{d}x$

Function space $H^1(I)$, $\tilde{H}^1(I)$ consists of functions u for which $\langle u, u \rangle_{H^1} < \infty$, $\langle u, u \rangle_{\tilde{H}^1} < \infty$, resp.

Remarks

- By using higher derivatives, spaces H^n and $\tilde{H}^n$ can be defined in a similar manner
- Similar definitions are possible without identification of endpoints or over $\mathbb{R}$

Sergey L. Sobolev

Sergey L'vovich Sobolev (1908–1989), studied function spaces and differential equations.
(Image: Russian Academy of Sciences, source: Wikipedia)

H^1 Gradients

Assume an energy functional E is given.

- By partial integration,

$$\langle u, v \rangle_{H^1} = \langle u, v - \lambda L^2 v'' \rangle_{H^0}$$

- Consequently, the H^1 gradient $v = \nabla_{H^1} E = \frac{\delta}{\delta_{H^1} u} E$ can be obtained as solution of the ODE

$$v - \lambda L^2 v'' = G$$

  where $G = \nabla_{H^0} E = \frac{\delta}{\delta_{H^0} u} E$ is the standard gradient

- Solution by convolution on the circle:

$$\nabla_{H^1} E = (K_\lambda * \nabla_{H^0} E)(x) = \int_0^L K_\lambda(y - x) \nabla_{H^0} E(y) \, \mathrm{d}y$$

  with kernel

$$K_\lambda(x) = \frac{\cosh \frac{x - L/2}{\sqrt{\lambda}\, L}}{2\sqrt{\lambda}\, L \sinh \frac{1}{2\sqrt{\lambda}}}, \qquad x \in [0, L]$$

  periodically extended to $\mathbb{R}$

$\tilde{H}^1$ Gradients

- By partial integration,

$$\langle u, v\rangle_{\tilde{H}^1} = \langle u, \bar{v} - \lambda L^2 v''\rangle_{H^0}$$

- Consequently, the $\tilde{H}^1$ gradient $v = \nabla_{\tilde{H}^1} E = \frac{\delta}{\delta_{\tilde{H}^1} u} E$ can be obtained as solution of the ODE

$$\bar{G} - \lambda L^2 v'' = G$$

with $\bar{v} = \bar{G}$, where G is the standard gradient as before

- Solution by convolution on the circle:

$$\nabla_{\tilde{H}^1} E = (\tilde{K}_\lambda * \nabla_{H^0} E)(x) = \int_0^L \tilde{K}_\lambda(y - x) \nabla_{H^0} E(y)\, \mathrm{d}y$$

with kernel

$$\tilde{K}_\lambda(x) = \frac{1}{L}\left(1 + \frac{(x/L)^2 - x/L + 1/6}{2\lambda}\right), \qquad x \in [0, L]$$

periodically extended to $\mathbb{R}$

Application to Curves

- Consider curve flow c_t
- The H^0 norm for the curve flow c_t penalises motion of curve points essentially separately
- No penalty for direction changes is involved, even singularities are ignored
- The Sobolev (H^1, $\tilde{H}^1$) norms for curve flows penalise direction changes, thereby favouring translational motion of c

Gradient Flows for Curve Length

- Consider a smooth closed curve $c : [0, L] \to \mathbb{R}^2$
- Compute H^1 gradient descent for curve length energy

$$E[c] = \oint_c \mathrm{d}s$$

- The H^0 gradient is conventional curvature motion

$$\nabla_{H^0} E = -L\kappa\vec{n}$$

Gradient Flows for Curve Length, cont.

- Using

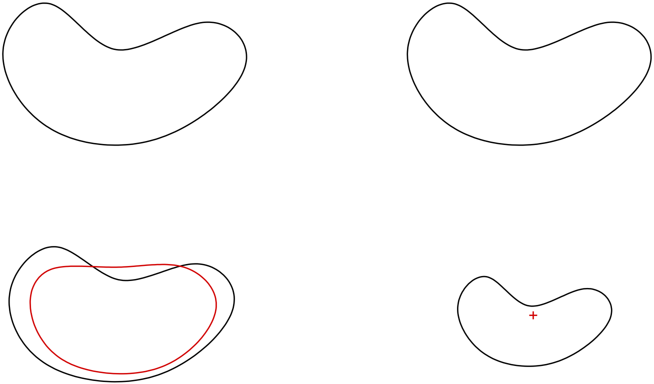
$$K_\lambda''(x) = \frac{1}{\lambda L^2}(K_\lambda - \delta(x)) \,, \qquad \tilde{K}_\lambda''(x) = \frac{1}{\lambda L^2}\left(\frac{1}{L} - \delta(x)\right)$$

  one computes

$$\nabla_{H^1} E = \frac{1}{\lambda L}(c - K_\lambda * c)$$
$$\nabla_{\tilde{H}^1} E = \frac{c - \bar{c}}{\lambda L}$$

- Gradient flows prevent the generation of singularities and are stable for gradient descent and even gradient ascent
- The $\tilde{H}^1$ gradient descent preserves the shape – it simply rescales the curve ($\bar{c}$ is nothing but the centre of gravity of the curve)
- The H^1 gradient descent simplifies the curve (note that the convolution $K_\lambda * c$ smoothes the curve coordinates)

Gradient Flows for Curve Length, cont.



Top left and right: Initial curve. **Bottom left:** Curve modified by H^1 gradient descent and **smoothed curve** $K_\lambda * c$ (schematic). **Bottom right:** Curve modified by $\tilde{H}^1$ gradient descent and **centre point** $\bar{c}$ (schematic).

References

General

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Affine differential geometry, affine curvature motion

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