Lecture 4

- Review of Level Sets
- Level Set Evolutions in the Plane
- Morphological Operations with Level Sets
- Algorithmic Aspects
- Curvature Motion on Level Sets

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Level Sets

- Consider smooth function $u : \Omega \to \mathbb{R}, \Omega \subseteq \mathbb{R}^2$ open
- Choose some number $z \in \mathbb{R}$
- The set
$$L_z(u) := \{(x, y) \in \Omega : \; u(x, y) = z\}$$
is called a level set of u
- Connected components of $L_z(u)$ are isolated points or curves

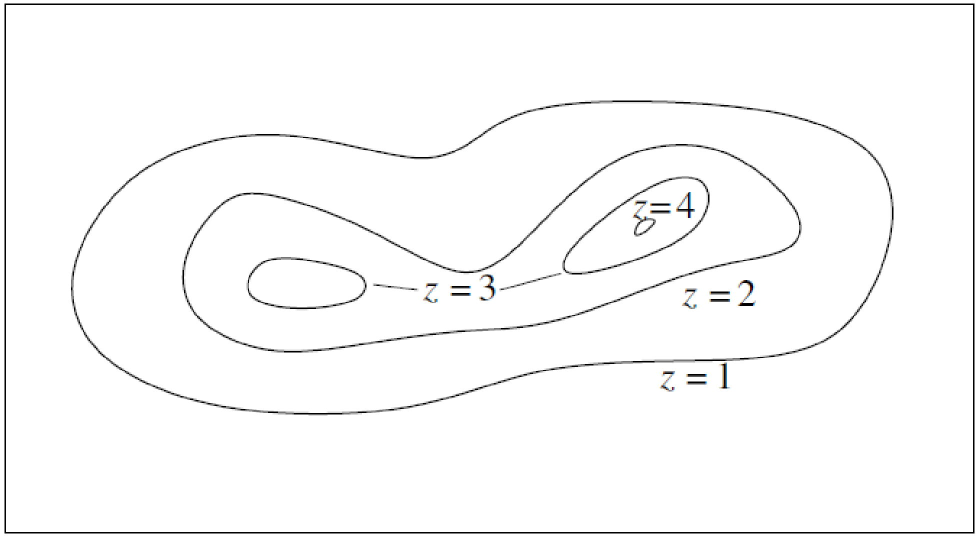

Figure: Four level sets of a function in the plane (schematic).

Parametrised Level Lines

- Consider u as before
- Consider a curve c which is a connected component of a level set $L_z(u)$
- Orientation convention. Let c be parametrised such that the smaller values of u lie on the left-hand side of c
- Equivalent:
  - The normal vector $\vec{n}$ points to the smaller values of u
  - The normal vector $\vec{n}$ and the gradient of u point in opposite directions

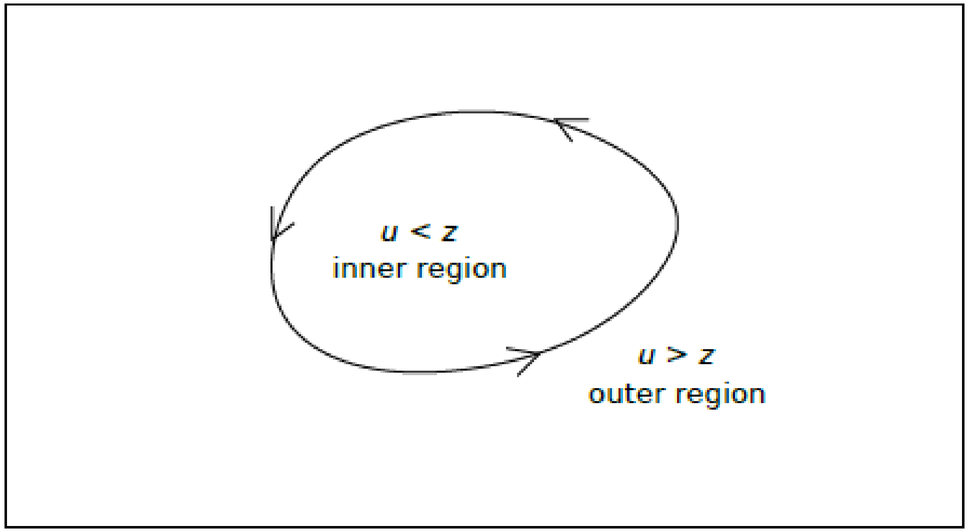

Figure: Orientation convention for level lines.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

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Level Sets Curve Equations

Relation of arc-length parametrisation of a level line and the partial derivatives of u :

- Let $c(s) = (x(s), y(s))^\top$, $u(c(s)) = z$, with $||c_s(s)|| = 1$. Then

$$||c_s(s)|| = x_s^2(s) + y_s^2(s) = 1,$$

$$\frac{du(c(s))}{ds} = \langle \nabla u, c_s \rangle = u_x x_s + u_y y_s = 0,$$

and

$$x_s(s) = \frac{-u_y}{\sqrt{u_x^2 + u_y^2}}, \quad y_s(s) = \frac{u_x}{\sqrt{u_x^2 + u_y^2}},$$

because of the orientation convention.

- By integration, the curve equations can be obtained:

$$x(s) = x(0) + \int_0^s x_s(\sigma)\, d\sigma \qquad y(s) = y(0) + \int_0^s y_s(\sigma)\, d\sigma$$

where (x_0, y_0) is a starting point belonging to the level set

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21 22
23

Signed Distance Function

- Let a sufficiently smooth closed regular curve c be given
- c separates the plane into an inner and an outer region
- To each point (x, y) in the plane, assign as $u(x, y)$
  - the distance of (x, y) to c if (x, y) is in the outer region
  - (-1) times the distance of (x, y) to c if (x, y) is in the inner region
  - 0 if (x, y) lies on c
- Then u is continuous, and u is differentiable within some band enclosing c
- u is called signed distance function of c
- c is the zero-level set $L_0(u)$

Remark: The construction is equally possible if a set of closed regular curves is given, with some compatibility condition on orientations, and allows then to construct a function u for which the union of the curves is the zero-level set

Curvature of a Level Line

- Consider function u and level line c in arc-length parametrisation
- Tangent vector: $\overrightarrow{t}(s) = (x_s, y_s)^T$
  Normal vector: $\overrightarrow{n}(s) = \overrightarrow{t}(s)^\perp = (-y_s, x_s)^T$
- Curvature definition $(x_{ss}, y_{ss})^T = \kappa \overrightarrow{n}$ implies

$$\kappa = -\frac{x_{ss}}{y_s} = \frac{y_{ss}}{x_s}.$$

- Evaluation gives

$$\kappa(c(s)) = -\frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{(u_x^2 + u_y^2)^{3/2}}$$

(derivation: next slide)

Curvature of a Level Line, Cont.

Sketch of the derivation:
Since $\nabla u \perp c_s$ we get that $x_s = -\frac{u_y}{||\nabla u||}, y_s = \frac{u_x}{||\nabla u||}$. Thus

$$\begin{aligned}
x_{ss}(c(s)) &= \langle \nabla x_s(c(s)), c_s(s) \rangle \\
&= -\frac{u_y}{||\nabla u||} \cdot \frac{u_{xy}(u_x^2+u_y^2) - u_y(u_x u_{xx} + u_y u_{xy})}{||\nabla u||^3} \\
&\quad + \frac{u_x}{||\nabla u||} \cdot \frac{u_{yy}(u_x^2+u_y^2) - u_y(u_x u_{xy} + u_y u_{yy})}{||\nabla u||^3} \\
&= \frac{1}{||\nabla u||^4}(-u_x^2 u_y u_{xy} - u_y^3 u_{xy} + u_x u_y^2 u_{xx} + u_y^3 u_{xy} \\
&\quad + u_x^3 u_{yy} + u_x u_y^2 u_{yy} - u_x^2 u_y u_{xy} - u_x u_y^2 u_{yy}) \\
&= \frac{u_x}{||\nabla u||^4}(u_y^2 u_{xx} - 2 u_x u_y u_{xy} + u_x^2 u_{yy})
\end{aligned}$$

Finally, since $c_{ss} = \kappa \overrightarrow{n}$,

$$\kappa = \frac{x_{ss}}{-y_s} = -\frac{u_x(u_y^2 u_{xx} - 2 u_x u_y u_{xy} + u_x^2 u_{yy})}{||\nabla u||^4} \cdot \frac{||\nabla u||}{u_x}$$

Supplement: Curvature in arbitrary parametrisation

Result. For a curve $c(p) = (x(p), y(p))$ we have that

$$k(p) = \frac{x_p y_{pp} - y_p x_{pp}}{||x_p^2 + y_p^2||^{3/2}}$$

Sketch of proof: Assume $\tilde{c} = c \circ \phi$ is parametrised by arc-length, then

$$\phi'(s) = ||c_p(\phi(s))||^{-1}, \qquad \phi''(s) = ||c_p(\phi(s))||^{-4} < c_p(\phi(s)), c_{pp}(\phi(s)) > .$$

It follows that

$$\begin{aligned} \tilde{c}_p &= c_p(\phi(s))||c_p(\phi(s))||^{-1} \\ \tilde{c}_{pp}(p) &= \frac{c_{pp}(\phi(s))}{||c_p(\phi(s))||^2} - \frac{< c_{pp}(\phi(s)), c_p(\phi(s)) >}{||c_p(\phi(s))||^4} c_p(\phi(s)). \end{aligned}$$

Evaluating $\kappa(s)^2 = \langle \tilde{c}_{pp}(s), \tilde{c}_{pp}(s) \rangle$ the result follows.

Level Set Evolutions

- Consider curve evolution $c(p,t) : I \times [0,T) \to \mathbb{R}^2$ of closed curve
- Let $\overrightarrow{t}$ tangent vector, $\overrightarrow{n}$ normal vector of c
- Consider smooth image evolution $u(x,y,t) : \Omega \times [0,T) \to \mathbb{R}$
- Assume $c(\cdot,t)$ is a level set (component) of $L_z(u(\cdot,\cdot,t))$ for each t, respecting our orientation convention
- Then for any fixed t (slide 4)

$$x_s(s,t) = \frac{-u_y}{\sqrt{u_x^2+u_y^2}}, \qquad y_s(s) = \frac{u_x}{\sqrt{u_x^2+u_y^2}},$$

thus

$$\overrightarrow{n} = -\frac{\nabla u}{||\nabla u||}.$$

- Then one speaks of a level set evolution

Correspondence between Level Line and Image Evolutions

Result. The following relation between image evolution and level line evolution holds:

$$\frac{\partial c}{\partial t} = \beta(c(p,t),t)\overrightarrow{n}(p,t) \qquad \Longleftrightarrow \qquad \frac{\partial u}{\partial t} = \beta(x,y,t) \cdot ||\nabla u|| \,.$$

Sketch of proof: For the level line $Lz(u(\cdot,t))$ we have $u(c(p,t),t) = z$ and time derivative gives:

$$u_x x_t + u_y y_t + u_t = \langle \nabla u, c_t \rangle + u_t = 0$$

Thus, the scalar product of ∇u and the curve evolution gives

$$\left\langle \nabla u, \frac{\partial c}{\partial t} \right\rangle = -u_t = \beta(c(p,t),t) \left\langle \nabla u, \overrightarrow{n}(p,t) \right\rangle .$$

Equivalently

$$0 = \beta \left\langle \nabla u, \overrightarrow{n} \right\rangle + u_t = \beta \cdot ||\nabla u|| - \partial_t u.$$

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Special Case: Signed Distance Functions

- Assume u is the signed distance function for c at time t (*)
- Then $||\nabla u|| = 1$
- Thus

$$\frac{\partial c}{\partial t} = \beta(c(p,t),t)\overrightarrow{n}(p,t) \qquad \Longleftrightarrow \qquad \frac{\partial u}{\partial t} = \beta(x,y,t)\,.$$

- Caveat: The property (*) is not preserved by the evolution
- Consequence: In applications, the signed distance function needs to be restored in each time step

Remark: Arc-Length Parametrisation

As with most curve flows, arc-length parametrisation is not preserved over time.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Dilation and Erosion Flows

- Consider erosion

$$c_t = \overrightarrow{n}$$

- Assume c is level line of image u
- Evolution of u: dilation PDE:

$$u_t = ||\nabla u|| \, .$$

- Similarly, one obtains the erosion PDE

$$u_t = -||\nabla u|| \, .$$

Application to a Shape

- Assume c is a curve representing a shape
- Make c into zero level set $L_0(u)$ of u by choosing u as signed distance function of c
- Apply evolution equation

$$u_t = \pm||\nabla u||$$

  to u
- Obtain evolved curve as zero level set of $u(t)$

Advantages of the Level Set Formulation for Dilation and Erosion

- Direct implementation of curve evolution is difficult:
  - Requires curve representation, typically by sampled points
  - Sampled points may be equally distributed at the begin, becomer thinner or denser in different regions during evolution need for resampling procedure
  - Topology changes due to singularities and self-intersections need to be handled specially

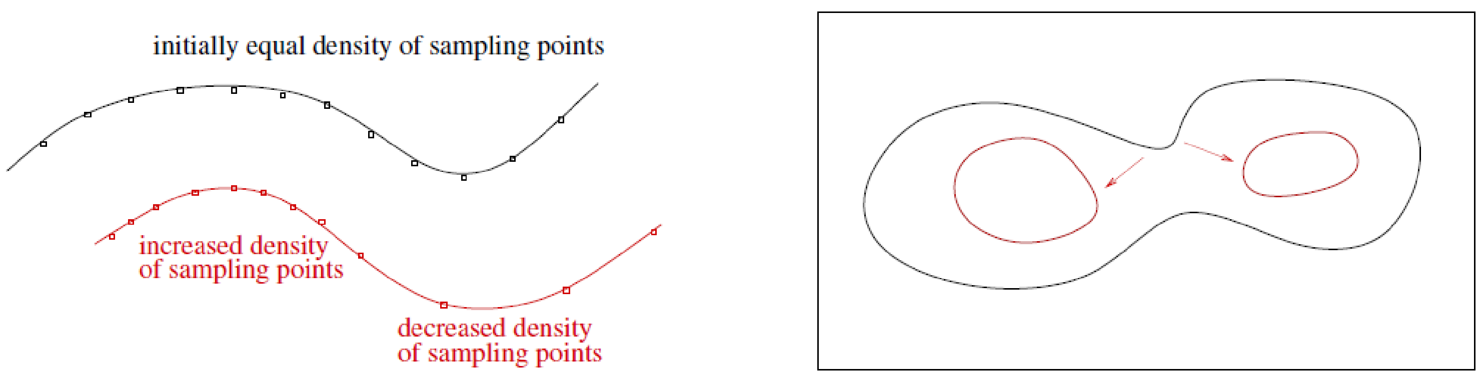


Figure: Problems in direct implementations of curve evolutions. **Left:** Change in sampling density requires re-sampling. **Right:** Topology changes may occur.

Advantages of the Level Set Formulation for Dilation and Erosion, cont.

- Level set formulation removes these problems
  - No explicit curve representation necessary, thus also no resampling
  - Topology changes are accounted for automatically

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Algorithmic Aspects

In the simplest case, an explicit time-stepping scheme is used to compute the evolution

$$u_t = \pm||\nabla u|| = \pm\sqrt{u_x^2 + u_y^2}$$

- Discretise u_t by forward differences

$$[u_t]_{i,j}^k = \frac{1}{\tau}(u_{i,j}^{k+1} - u_{i,j}^k)$$

  in pixel (i,j) and time step k, where τ is the time step size
  (Notice: We use $[\cdot]$ to denote approximation)

- Discretise u_x and u_y on the right-hand side by central differences

$$[u_x]_{i,j}^k = \frac{1}{2h_x}(u_{i+1,j} - u_{i-1,j})$$

$$[u_y]_{i,j}^k = \frac{1}{2h_y}(u_{i,j+1} - u_{i,j-1})$$

  with spatial step sizes h_x, h_y

Algorithmic Aspects, Cont.

Compute values of new time step:

$$u_{i,j}^{k+1} = u_{i,j}^{k} \pm \tau \sqrt{([u_x]_{i,j}^{k})^2 + ([u_y]_{i,j}^{k})^2}$$

However, in the case of dilation and erosion, this discretisation tends to be instable.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Algorithmic Aspects, Cont.

To overcome numeric problems, an Upwind Scheme can be used.

- Discretise u_t as before
- Discretise u_x and u_y by one-sided differences dependent on the direction of the gradient. For dilation:
  - If $u_x > 0$, then use the forward difference approximation

$$[u_x]_{i,j}^k = \frac{1}{h_x}(u_{i+1,j}^k - u_{i,j}^k)$$

  - If $u_x < 0$, then use the forward difference approximation

$$[u_x]_{i,j}^k = \frac{1}{h_x}(u_{i,j}^k - u_{i-1,j}^k)$$

  - Analogously for u_y
- The explicit step to compute $u_{i,j}^{k+1}$ is as before, except that the new approximations for u_x, u_y are used.

Algorithmic Aspects, Cont.

Remarks.

- The upwind scheme is only useful for specific curve evolution processes (of hyperbolic type); dilation and erosion are of this type, while curvature flow is not of this type.
- The statements above describe just the idea. To make it precise, it is necessary to cover the case when forward and backward difference have opposite signs. For dilation this can be done, e.g., by

$$[|u_x|]_{i,j}^k = \frac{1}{h_x}(\max\{u_{i+1,j}^k, u_{i-1,j}^k, u_{i,j}^k\} - u_{i,j}^k)$$

- For erosion, the roles of the two approximations are swapped.

Application to Images

- Up to now, only the zero-level set of u was meaningful so we have used the PDEs only to evolve the zero-level set
- In fact, the PDEs evolve all level sets simultaneously
- Can therefore evolve entire grey-level images
- Then all level sets are dilated/eroded simultaneously
- Inclusion of level sets is preserved

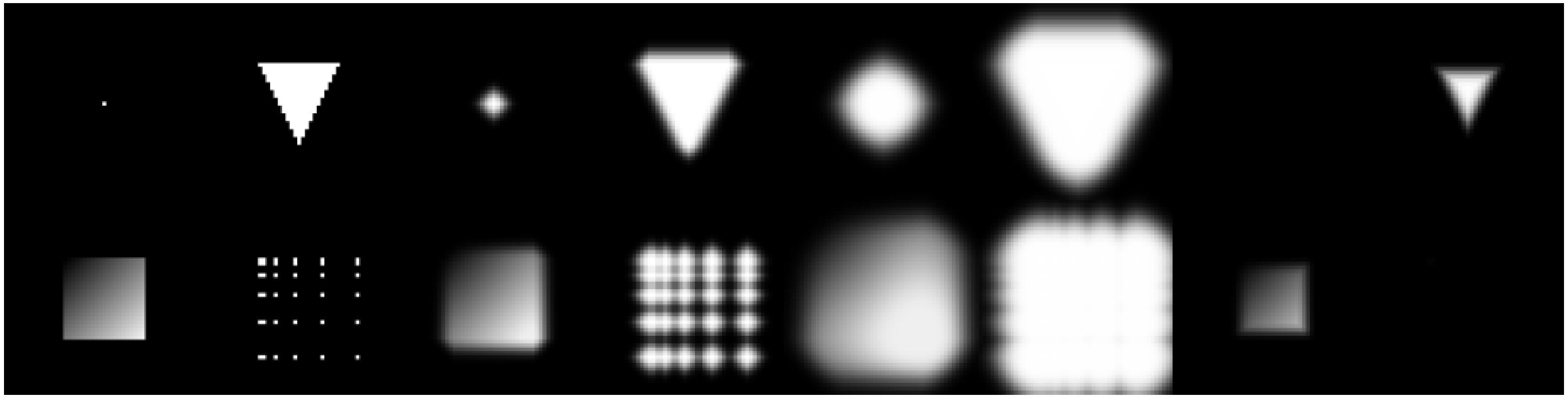


Figure: Left to right: Synthetic image, images after 10 and 40 iterations of dilation $u_t = ||\nabla u||$, after 10 iterations of erosion $u_t = -||\nabla u||$; in all cases $\tau = 0.25$

Application to Images, cont.

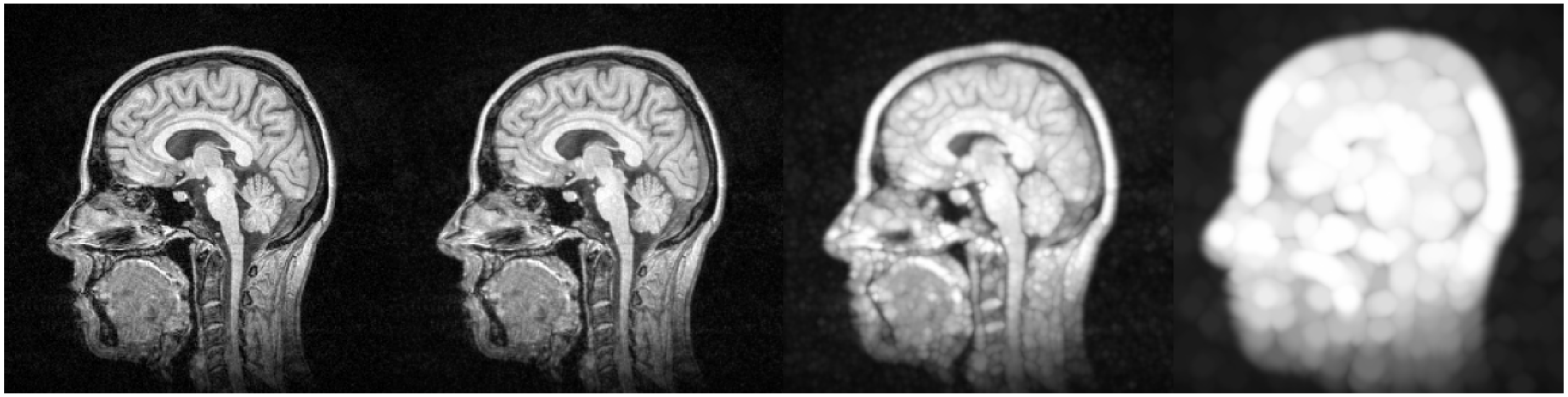


Figure: Left to right: Original MR image, images after 1, 10, 40 iterations at $\tau = 0.25$ of dilation

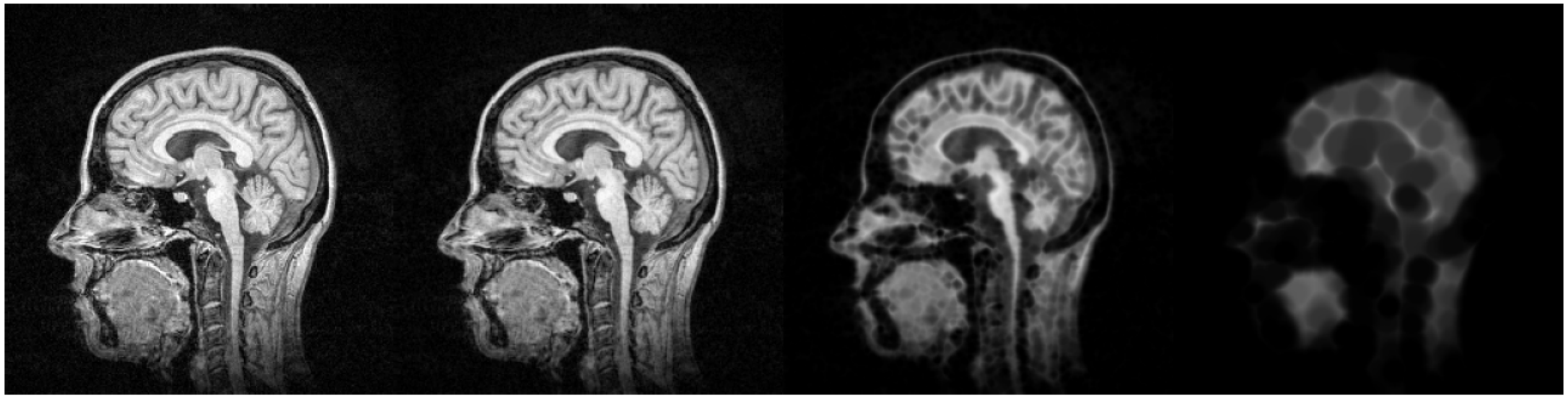


Figure: Left to right: Original MR image, images after 1, 10, 40 iterations at $\tau = 0.25$ of erosion

Curvature Motion

- Curve evolution

$$c_t = \kappa \overrightarrow{n}$$

corresponds to image evolution

$$u_t = \kappa \cdot ||\nabla u|| = -\frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2 + u_y^2}$$

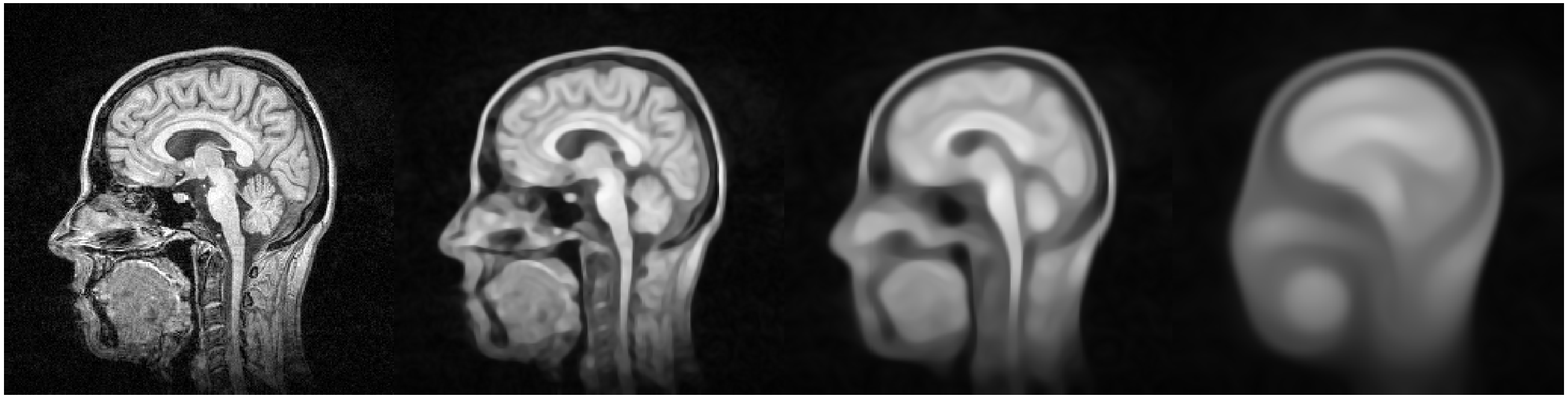


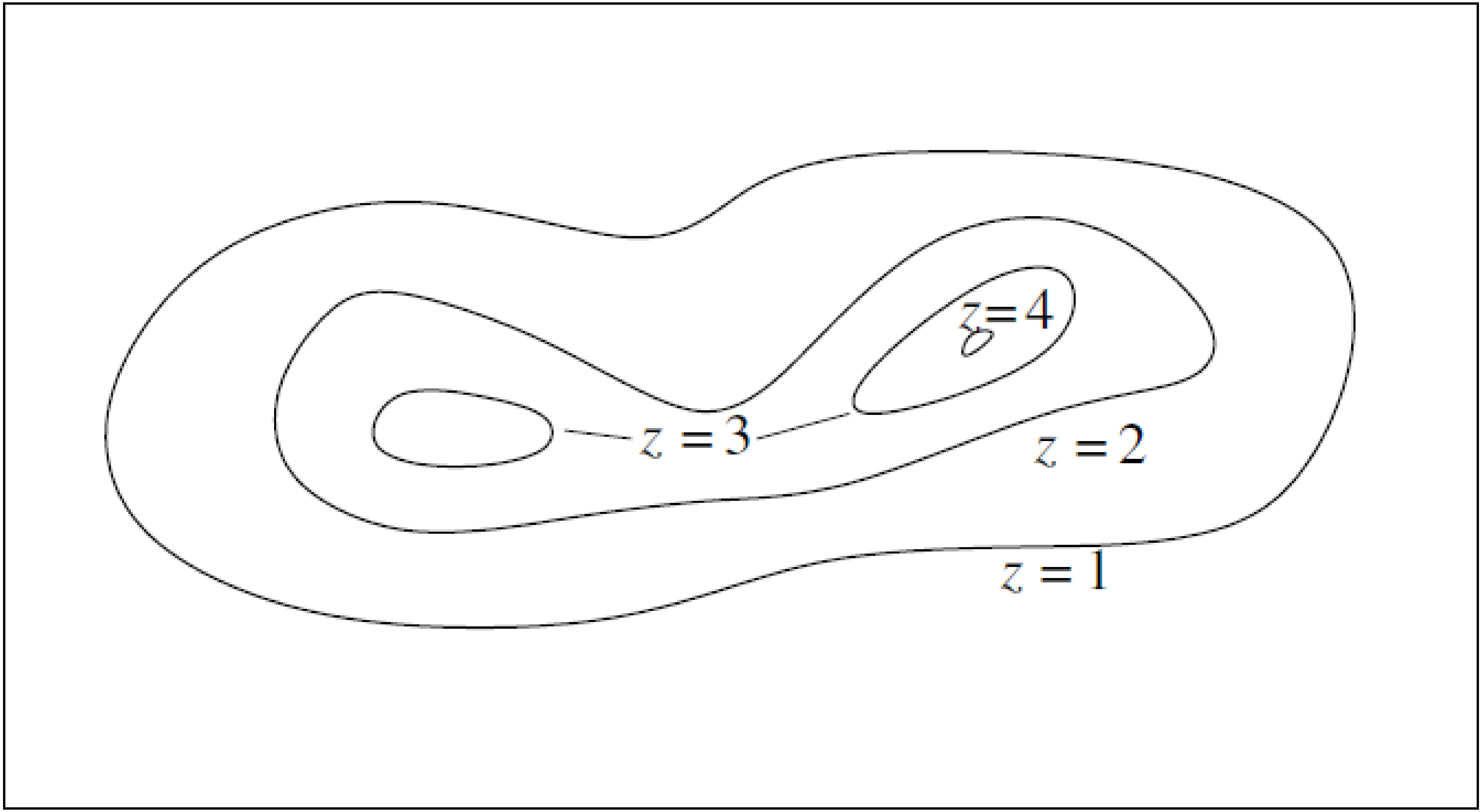
Figure: Left to right: Original MR image, images at evolution times 8, 80 and 800 of curvature motion $\beta = \kappa (\tau = 0.08)$

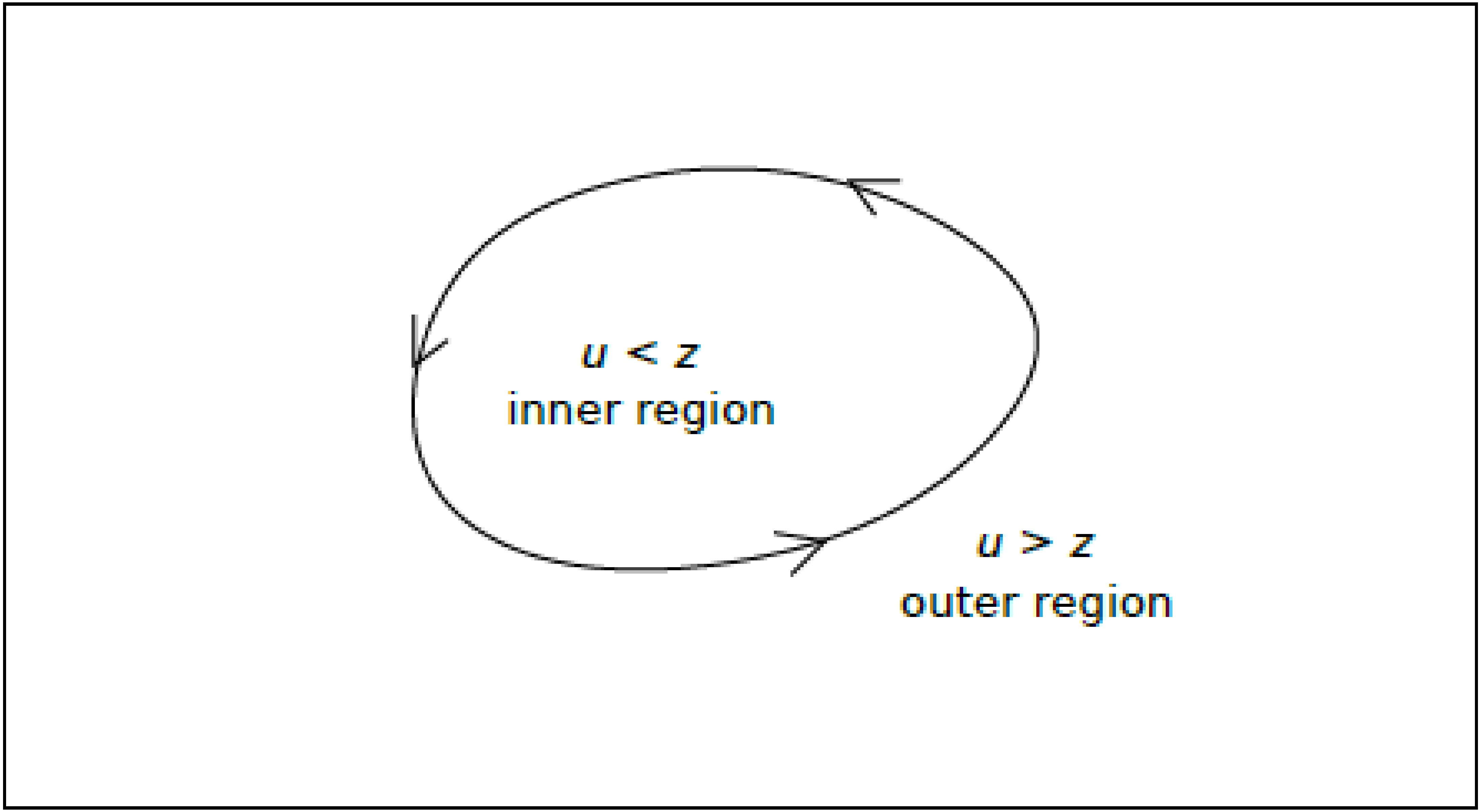
Remarks on Curvature Motion

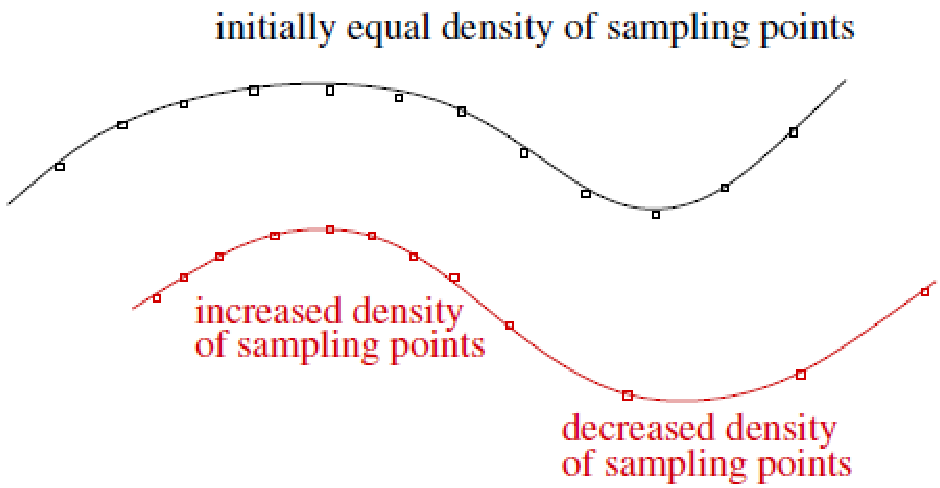
- Dilation, erosion and curvature motion of grey-value images share a remarkable property:

  Morphological Invariance. Dilation, erosion and curvature motion of grey-value images are invariant under arbitrary strictly monotonic rescalings of grey-values. The evolution of each level set does not depend on that of the other level sets.

- We will see a larger class of image filters sharing this property


z=4
z = 3
z = 2
z = 1


$u < z$
inner region
$u > z$
outer region


initially equal density of sampling points
increased density
of sampling points
decreased density
of sampling points