Lecture 3: Curve Evolutions in the Plane

- Curve Evolutions
- Morphological Operations
- Curvature Motion
- Level Sets in the Plane

Curvature

- Let $\vec{t}(s), \vec{n}(s)$ be unit vectors tangential and normal to c at $c(s)$, resp., and $(\vec{t}, \vec{n})$ positively oriented
- Then
$$c_s(s) = \vec{t}(s) \quad c_{ss}(s) = \kappa(s)\vec{n}(s)$$
with a uniquely determined function $\kappa(s)$
- $\kappa(s)$ is called curvature of c at $c(s)$

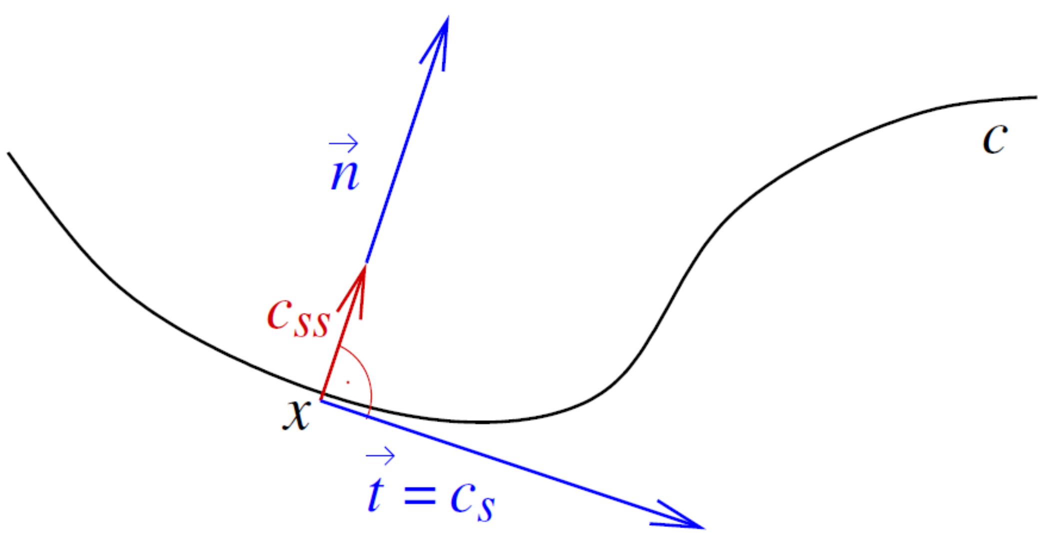


Figure: Curve c with tangent and normal vectors, first and second derivatives at point $x = c(s)$.

Curve Evolutions in $\mathbb{R}^d$

- Consider curves parametrised by interval $I \subseteq \mathbb{R}$
- Introduce additional time parameter $t \in [0, T]$, $T \geq 0$
- Curve evolution: differentiable function $c : I \times [0, T] \to \mathbb{R}^d$
  - For each fixed t, $c(\cdot, t)$ is a curve
  - Initial curve: $c_0(p) = c(p, 0)$
  - For fixed p, $c(p, \cdot)$ is a trajectory of a curve point
  - Time derivative c_t is called curve flow

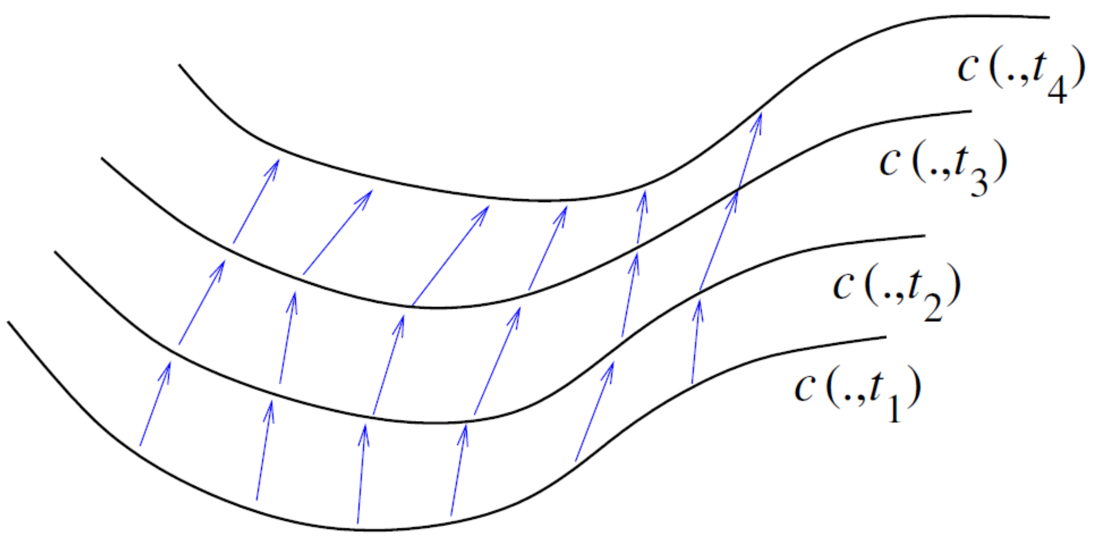


Figure: Curve evolution, $t1 < t2 < t3 < t4$

Decomposition of Planar Flows

Consider a curve evolution $c : I \times [0, T] \to \mathbb{R}^2$.

- Write time evolution of a curve point in terms of tangential and normal vectors

$$\frac{\partial c(p,t)}{\partial t} = \alpha(p,t)\overrightarrow{t}(p,t) + \beta(p,t)\overrightarrow{n}(p,t),$$
$$c(p,0) = c_0(p)$$

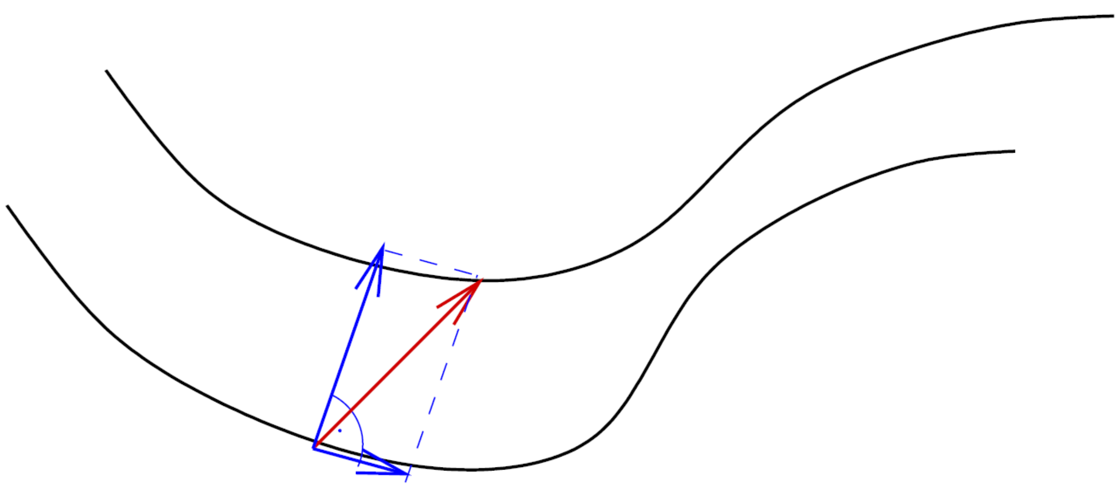


Figure: Decomposition of curve flow into tangential and normal components

Role of the Normal Flow

- Curve evolution c in $\mathbb{R}^2$
- Assume that the normal velocity $\beta(p,t) = \tilde{\beta}(x,t)$ depends only on $x = c(p,t)$ and t
- Then the curve evolution $\tilde{c}$ given by

$$\frac{\partial \tilde{c}(p,t)}{\partial t} = \beta(\tilde{c}(p,t),t)\overrightarrow{n}(\tilde{c},t)$$

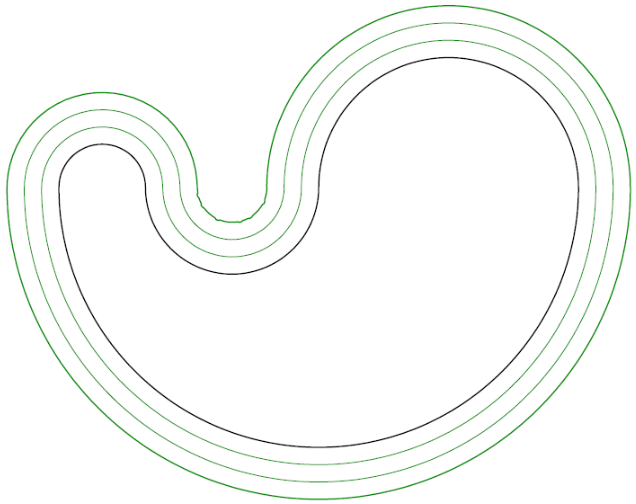
describes the same family of curves, i.e. $\tilde{c}(\cdot,t)$ is a reparametrisation of $c(\cdot,t)$ for each t

- Particularly: A flow $c_t = \alpha \overrightarrow{t}$ does not change the shape of a closed curve c. The normal flow is what governs the shape evolution

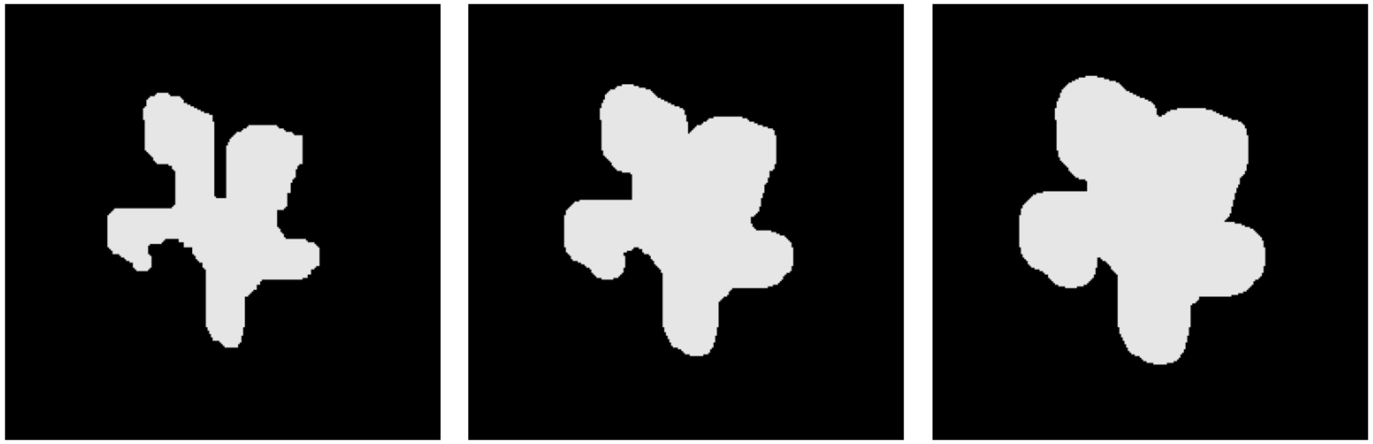
Dilation of Curves

- Closed simple regular curve c with $\oint_c \kappa(s)ds = 2\pi$ (rotation number 1)
- Evolution
$$c_t = -\overrightarrow{n}(c,t) \qquad \text{(i.e. } \beta = -1)$$
moves all curve points at equal velocity in global outward direction
- If the initial curve $(t = 0)$ has a lower bound $-K < 0$ for the curvature, this yields a differentiable curve evolution for $t \in [0, \frac{1}{K}[$. For larger t, singularities (e.g. cusps) may occur. Self-intersections may even occur earlier!
- If the closed curve is considered as object shape, then this process describes a dilation of this shape, uniformly enlarging the shape
- Problematic singularities: to continue the curve evolution, one might have to consider segments of the curve, even admit changes of topology (e.g. splitting into several connected components). Later we will discuss alternative descriptions that allow an easier handling of these phenomena.

Dilation of Curves



Dilation. **Black:** original curve, **green:** three time steps of curve evolution



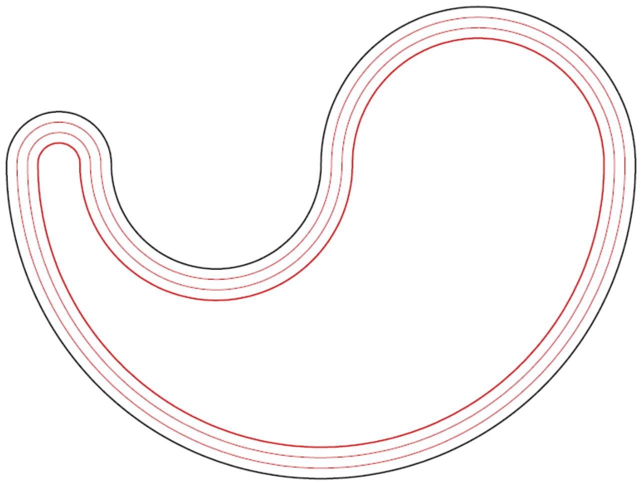
Dilation of a shape. **Left to right:** Original shape and two steps of dilation

Erosion of Curves

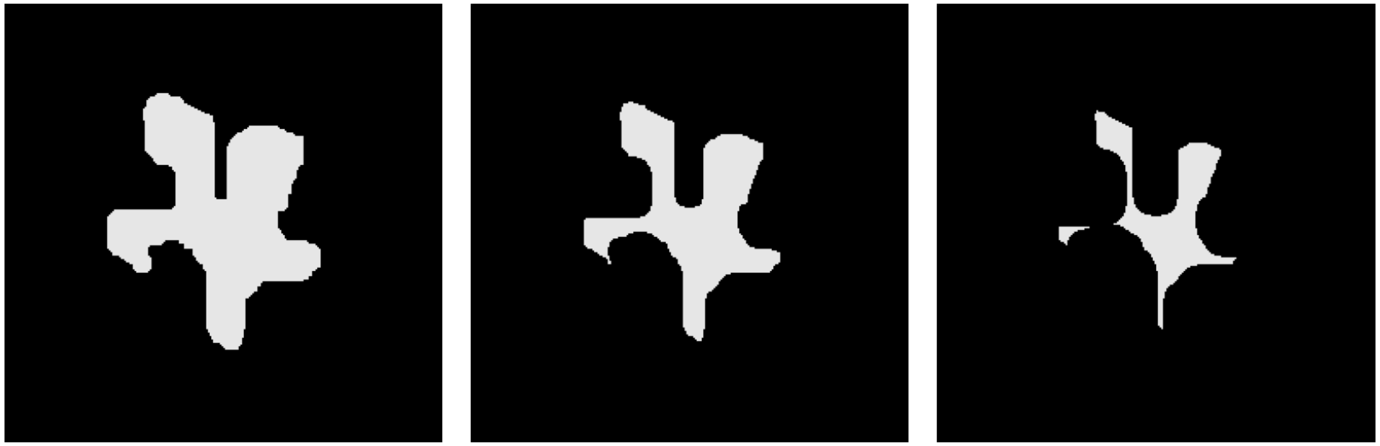
- Consider closed curve as before
- The evolution

$$c_t = \overrightarrow{n}(c, t) \qquad (i.e.\ \beta = +1)$$

  moves all curve points in global inward direction
- If $K > 0$ is upper bound for the curvature at $t = 0$, a differentiable curve evolution for $t \in [0, \frac{1}{K}[$ results
- Process describes erosion of object shapes, which chips away uniformly from the shape
- Problems with singularities similar as for dilation



Erosion. **Black:** original curve, **red:** three time steps of curve evolution



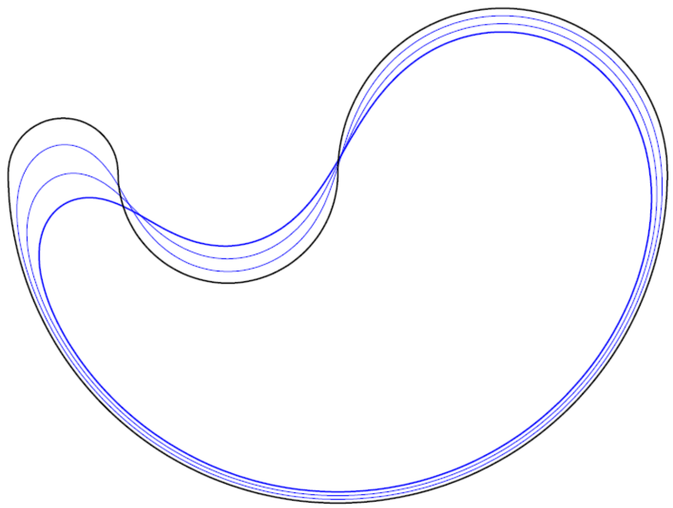
Erosion of a shape. **Left to right:** Original shape and two steps of erosion

Curvature Flow

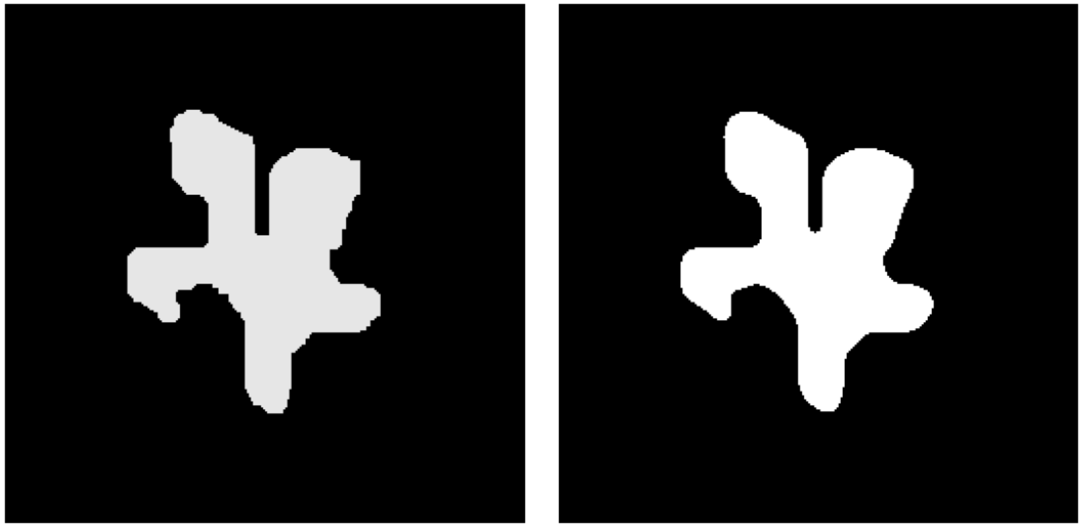
- Consider a curve in $\mathbb{R}^2$ with continuous curvature
- Curvature flow

$$c_t = \kappa(p,t)\overrightarrow{n}(c,t)$$

  moves curve in local inward direction at velocity given by the curvature
- Curvature flow is a shape-simplifying process
- Problems caused by singularities
- Other names: curvature motion, curve shortening flow, mean curvature flow/motion, geometric heat flow, geometric diffusion. Reasons for some of these names will become clear in later lectures



Curvature flow. **Black:** original curve, blue: three evolved curves at progressive times.



Curvature motion of the contour of a shape. **Left to right:** Original shape and processed version.

Properties of Curvature Flow

- circles remain circles
- connected closed curves remain connected
- The total absolute curvature $\oint_c |\kappa| ds$ of the regular curve c decreases monotonically under curvature motion.
  - it is 2π for convex curves
  - measures how far the curve is from being convex
- connected closed curves become convex after some time
- convex curves shrink to points in finite time
- curvature motion preserves the inclusion of curves
- The numbers of local maxima of κ and inflection points of c (where κ changes sign) decrease monotonically under curvature motion.

Properties of Curvature Flow

- The length $L(c)$ of a closed regular curve c decreases monotonically under curvature motion,

$$\frac{d}{dt}L(c) = -\int_c \kappa^2 \, ds$$

- The area $A(c)$ enclosed by a simple closed curve c decreases monotonically under curvature motion,

$$\frac{d}{dt}A(c) = -2\pi$$

Level Sets

- consider smooth function $u : \Omega \to \mathbb{R}, \Omega \subseteq \mathbb{R}^2$ open
- choose some value $z \in \mathbb{R}$
- The set
$$L_z(u) := \{(x, y) \in \Omega : \; u(x, y) = z\}$$
is called a level set of u.
- connected components of $L_z(u)$ are isolated points or curves

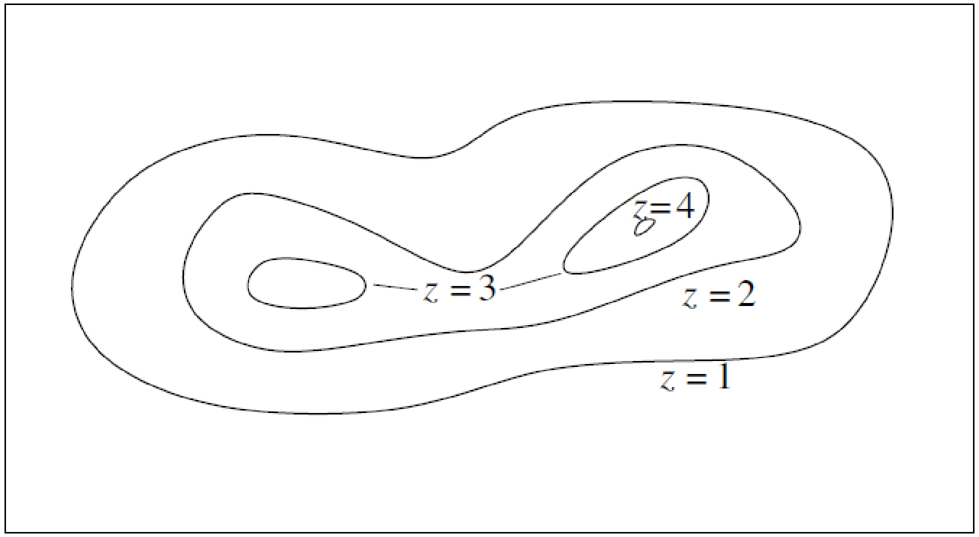


Figure: Four level sets of a function in the plane (schematic).

Level Sets in the Plane (2)

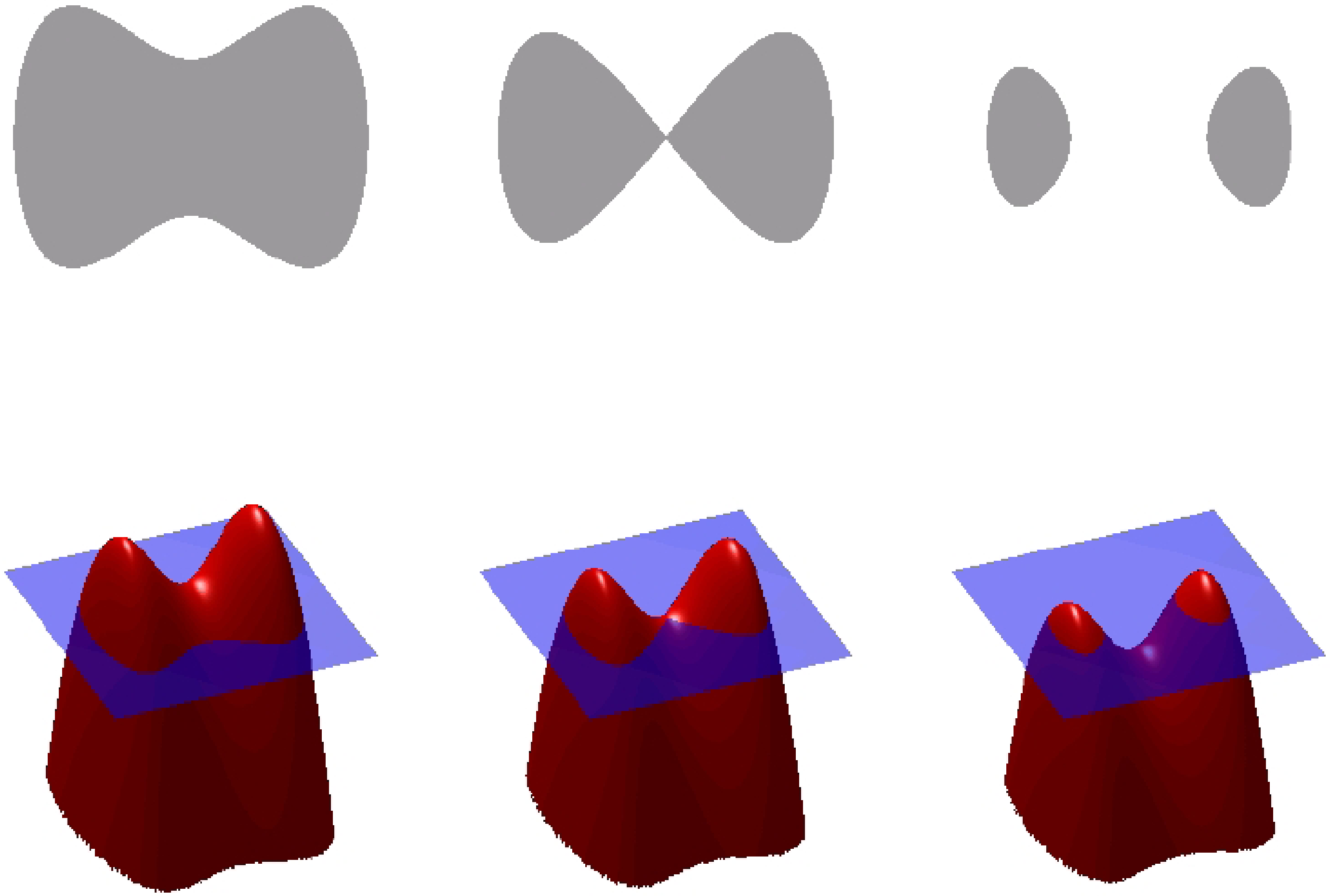


Figure: Level sets of a function over $\mathbb{R}^2$. In areas displayed in grey, function values are larger than z, i.e., the boundary of the grey areas constitutes the level set as defined here. Image: O. Alexandrov, from Wikipedia

Parametrised Level Lines

- consider u as before
- consider a curve c which is a connected component of a level set $L_z(u)$
- orientation convention: c is parametrised such that the smaller values of u lie on the left-hand side of c
- equivalent:
  - The normal vector $\vec{n}$ points to the smaller values of u.
  - The normal vector $\vec{n}$ and the gradient of u point in opposite directions.

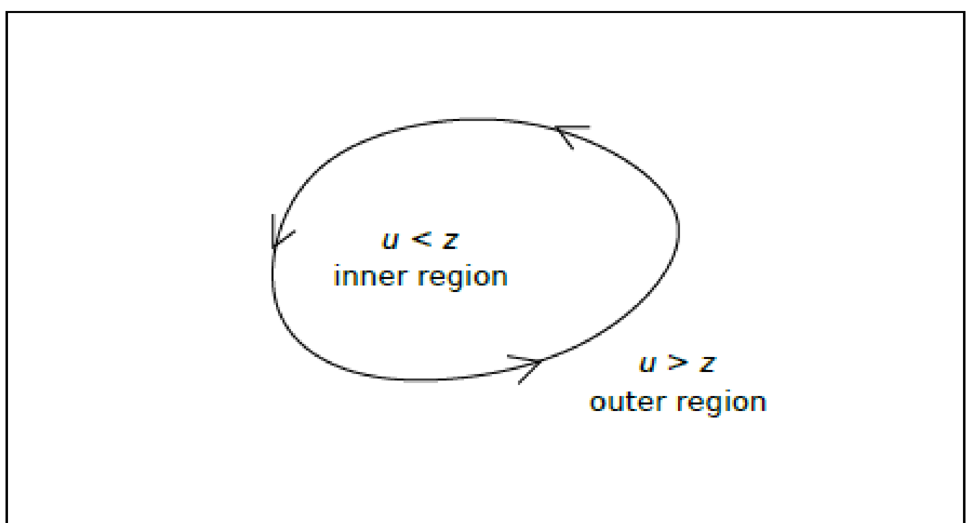


Figure: Orientation convention for level lines.

Derivation of Curve Equations

- Level line in arc-length parametrisation:

$$c(s) = (x(s), y(s))^\top, \qquad u(c(s)) = z, \qquad ||c_s(s)|| = 1.$$

- This implies

$$||c_s(s)|| = x_s^2(s) + y_s^2(s) = 1,$$

$$\frac{du(c(s))}{ds} = \langle \nabla u, c_s \rangle = u_x x_s + u_y y_s = 0,$$

$$x_s(s) = \frac{-u_y}{\sqrt{u_x^2 + u_y^2}} \qquad y_s(s) = \frac{u_x}{\sqrt{u_x^2 + u_y^2}}$$

- By integration, the curve equations can be obtained:

$$x(s) = x(0) + \int_0^s x_s(\sigma)\, d\sigma \qquad y(s) = y(0) + \int_0^s y_s(\sigma)\, d\sigma$$

where (x_0, y_0) is a starting point belonging to the level set

Signed Distance Function

- Let a sufficiently smooth closed regular curve c be given
- c separates the plane into an inner and an outer region
- To each point (x, y) in the plane, assign as $u(x, y)$
  - the distance of (x, y) to c if (x, y) is in the outer region
  - (-1) times the distance of (x, y) to c if (x, y) is in the inner region
  - 0 if (x, y) lies on c
- Then u is continuous, and u is differentiable within some band enclosing c
- u is called signed distance function of c
- c is the zero-level set $L_0(u)$

Remark: The construction is equally possible if a set of closed regular curves is given, with some compatibility condition on orientations, and allows then to construct a function u for which the union of the curves is the zero-level set

Curvature of a Level Line

- Consider function u and level line c in arc-length parametrisation
- tangent vector: $\overrightarrow{t}(s) = (x_s, y_s)^\top$
  normal vector: $\overrightarrow{n}(s) = \overrightarrow{t}(s)^\perp = (-y_s, x_s)^\top$
- Curvature definition $(x_{ss}, y_{ss})^\top = \kappa \overrightarrow{n}$ implies

$$\kappa = -\frac{x_{ss}}{y_s} = \frac{y_{ss}}{x_s}.$$

- Evaluation gives

$$\kappa(c(s)) = \frac{u_y^2 u_{xx} - u_x u_y u_{xy} + u_x^2 u_{yy}}{(u_x^2 + u_y^2)^{2/3}}$$

Level Set Evolutions

- Consider curve evolution $c(p, t) : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ of closed curve
- Let $\overrightarrow{t}$ tangent vector, $\overrightarrow{n}$ normal vector of c
- Consider smooth image evolution $u(x, y, t) : \Omega \times [0, T) \to \mathbb{R}$
- Assume $c(\cdot, t)$ is a level set (component) of $L_z(u(\cdot, \cdot, t))$ for each t, respecting our orientation convention
- Then

$$\overrightarrow{n} = -\frac{\nabla u}{||\nabla u||}.$$

- Then one speaks of a level set evolution

Correspondence between Level Line and Image Evolutions

- Characterisation of level line $Lz(u(\cdot, t))$ at time t :

$$u(c(p,t),t) = z \qquad \text{for all} \quad p$$

- Time derivative:

$$\begin{aligned} u_x x_t + u_y y_t + u_t &= 0 \\ \langle \nabla u, c_t \rangle + u_t &= 0 \end{aligned}$$

Correspondence between Level Line and Image Evolutions, cont.

- Curve evolution

$$\frac{\partial c}{\partial t} = \beta(c(p,t),t)\overrightarrow{n}(p,t)$$

equivalent to

$$\begin{aligned} 0 =& \beta \left\langle \nabla u, \overrightarrow{n} \right\rangle + u_t \\ =& - \frac{\beta}{||\nabla u||} \left\langle \nabla u, \nabla u \right\rangle + u_t \\ =& - \beta \cdot ||\nabla u|| + u_t \end{aligned}$$

and thus

$$\frac{\partial u}{\partial t} = \beta \cdot ||\nabla u|| \, .$$

- Result. Relation between image evolution and level line evolution:

$$\frac{\partial c}{\partial t} = \beta(c(p,t),t)\overrightarrow{n}(p,t) \qquad \Longleftrightarrow \qquad \frac{\partial u}{\partial t} = \beta(x,y,t) \cdot ||\nabla u|| \, .$$

Correspondence between Level Line and Image Evolutions, cont.

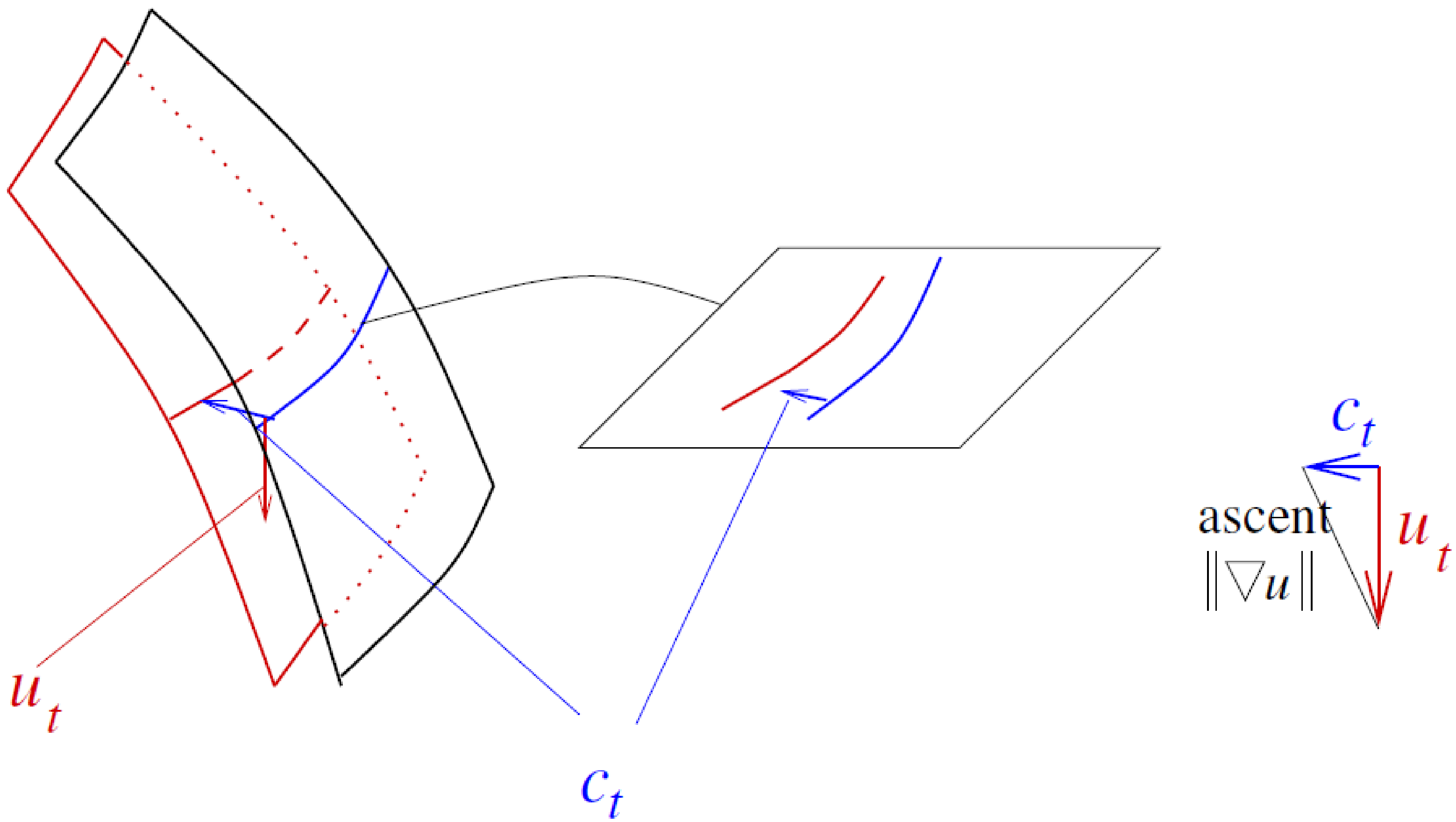


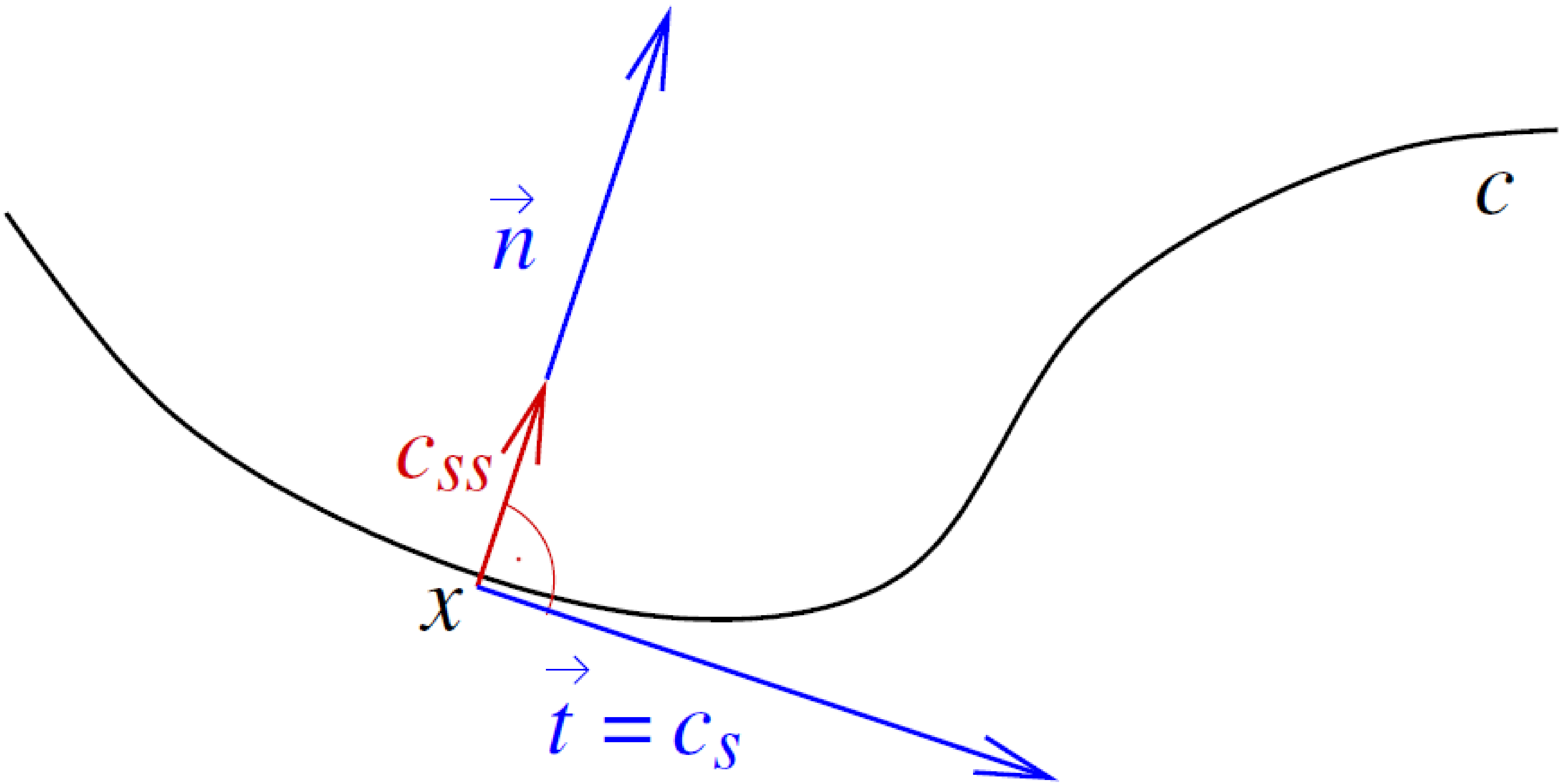
Figure: The relation between level line and image evolution.

Special Case: Signed Distance Functions

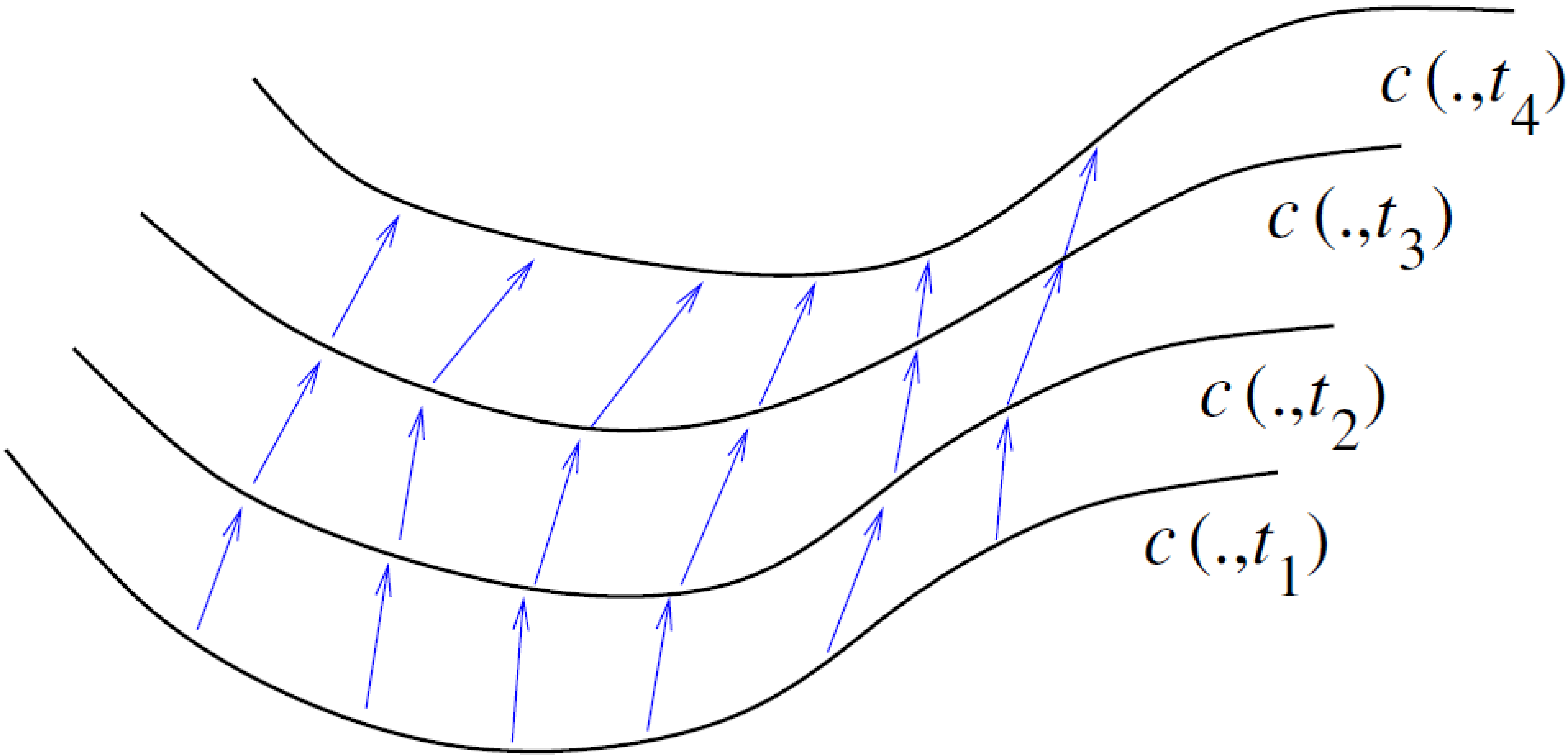
- Assume u is signed distance function for c at time t (*)
- Then $||\nabla u|| = 1$
- Thus
$$\frac{\partial c}{\partial t} = \beta(c(p,t),t)\overrightarrow{n}(p,t) \quad \Longleftrightarrow \quad \frac{\partial u}{\partial t} = \beta(x,y,t)\,.$$
- Caveat: The property (*) is not preserved by the evolution
- Consequence: In applications, the signed distance function needs to be restored in each time step

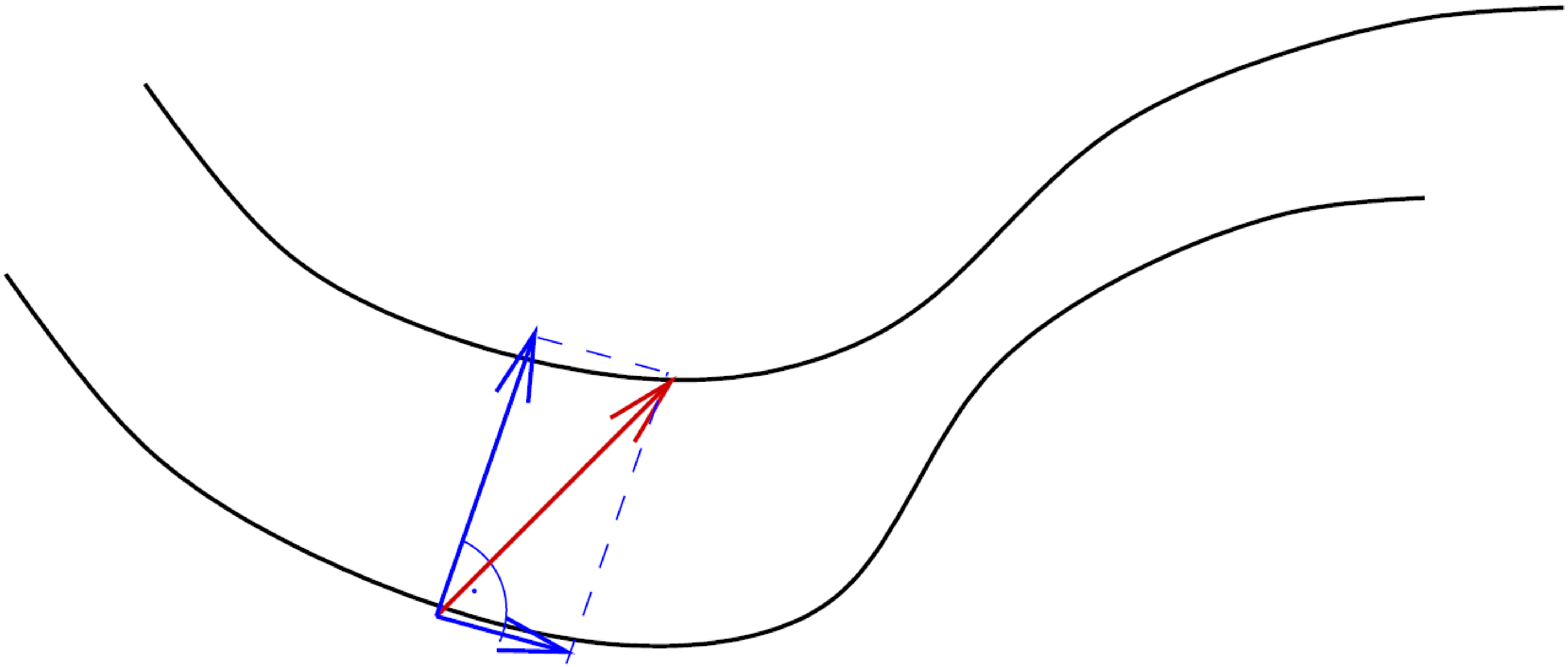
Remark: Arc-Length Parametrisation

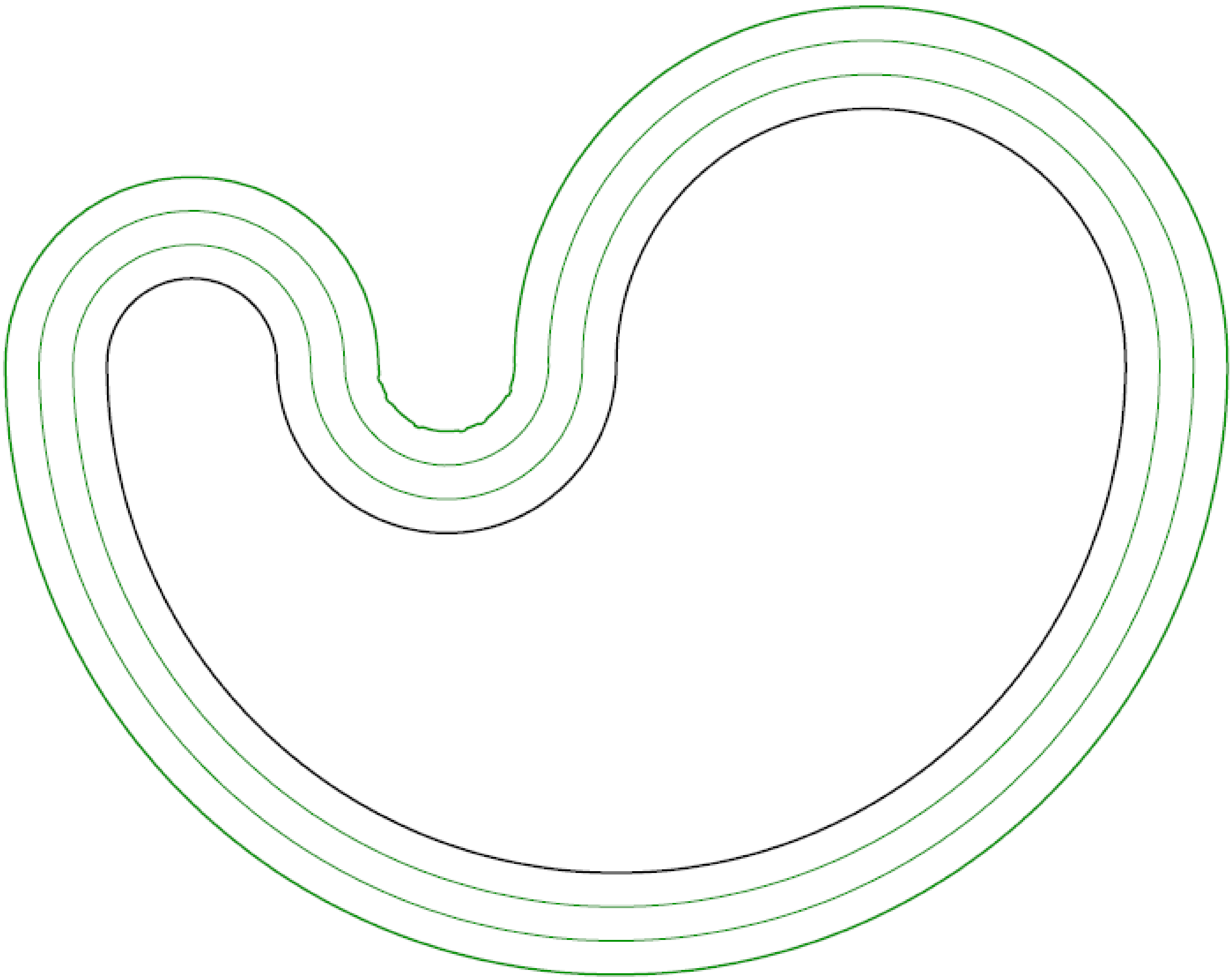
As with most curve flows, arc-length parametrisation is not preserved over time.

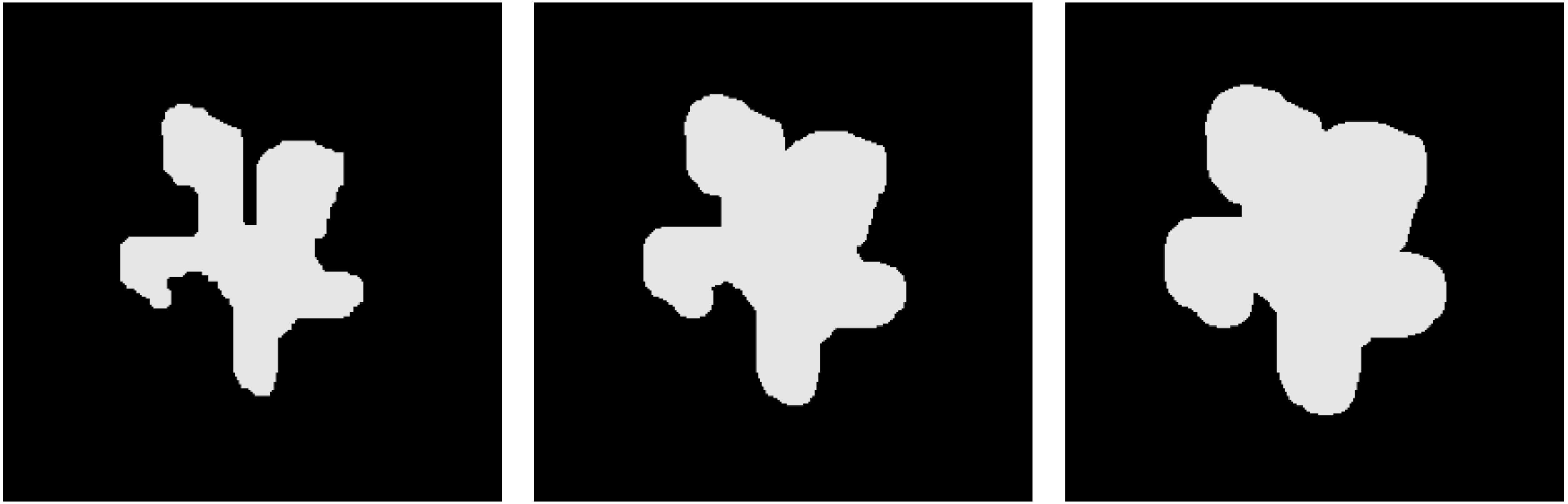


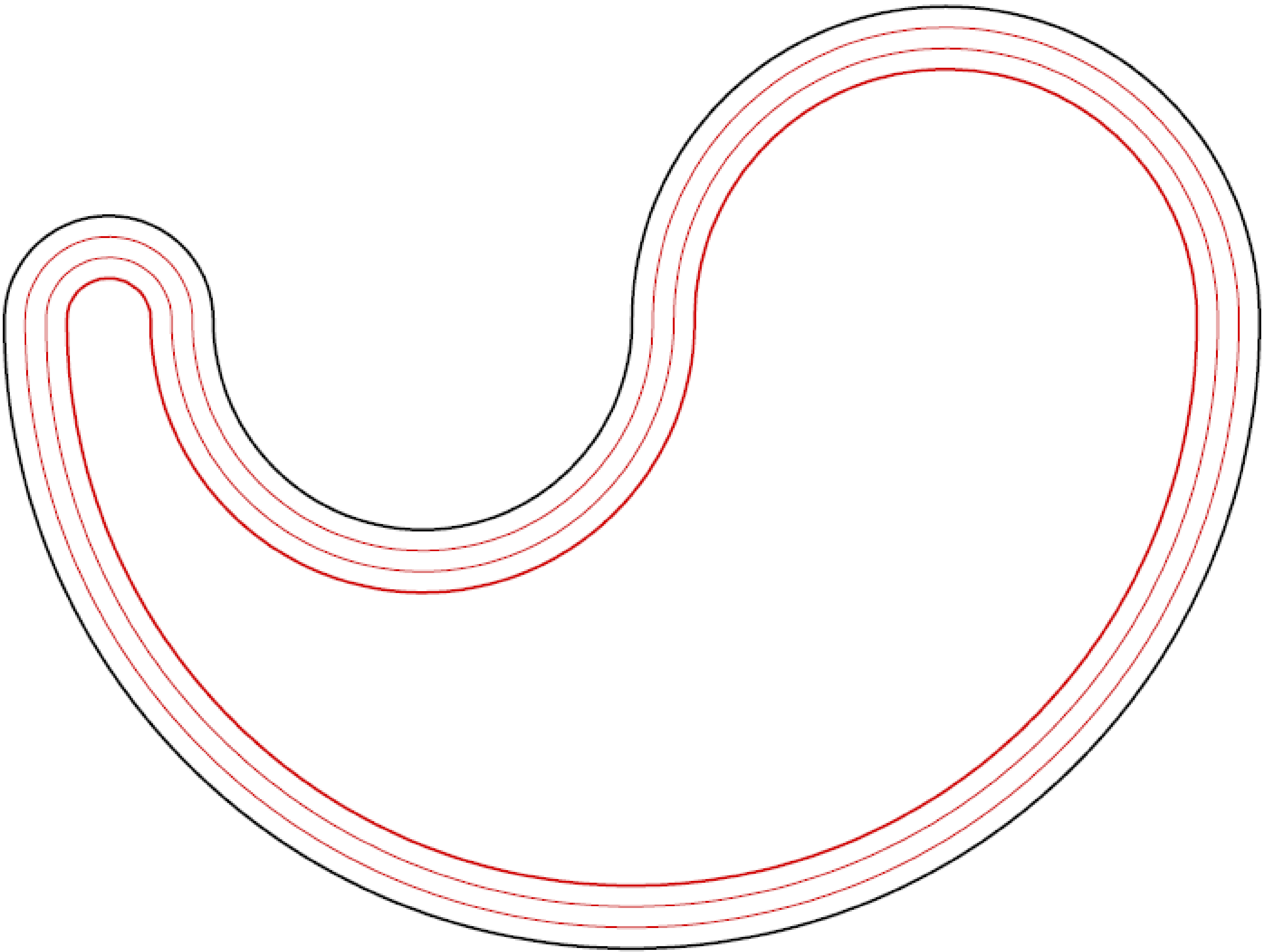
$\vec{n}$
c
c_{ss}
x
$\vec{t} = c_s$

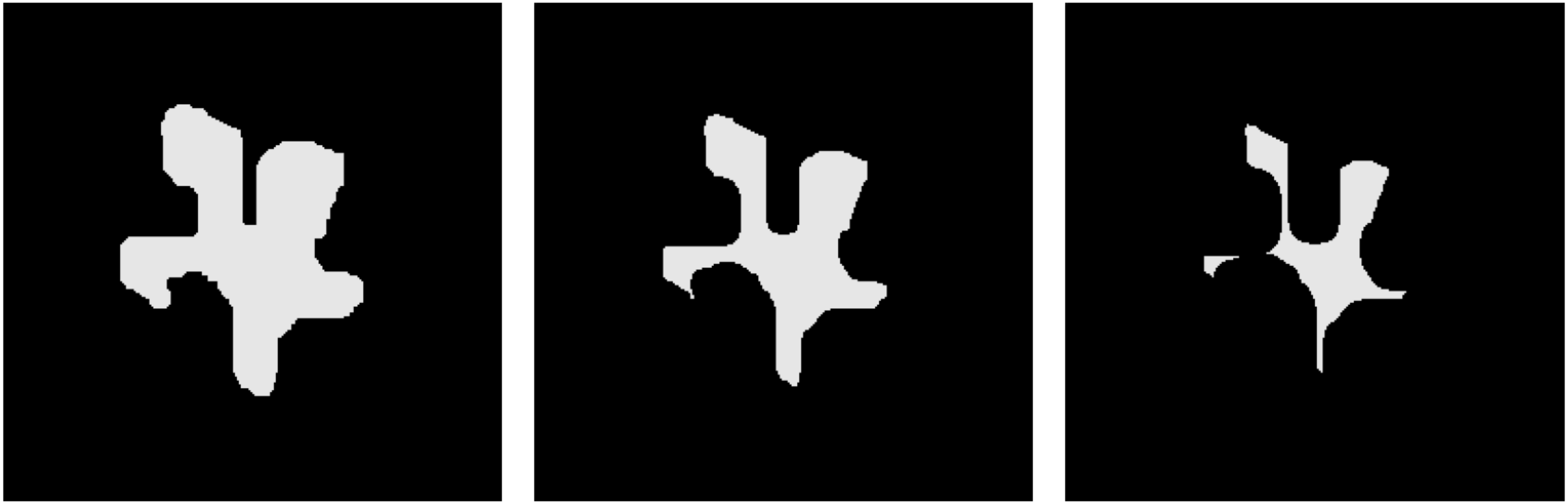

$c(.,t_4)$
$c(.,t_3)$
$c(.,t_2)$
$c(.,t_1)$

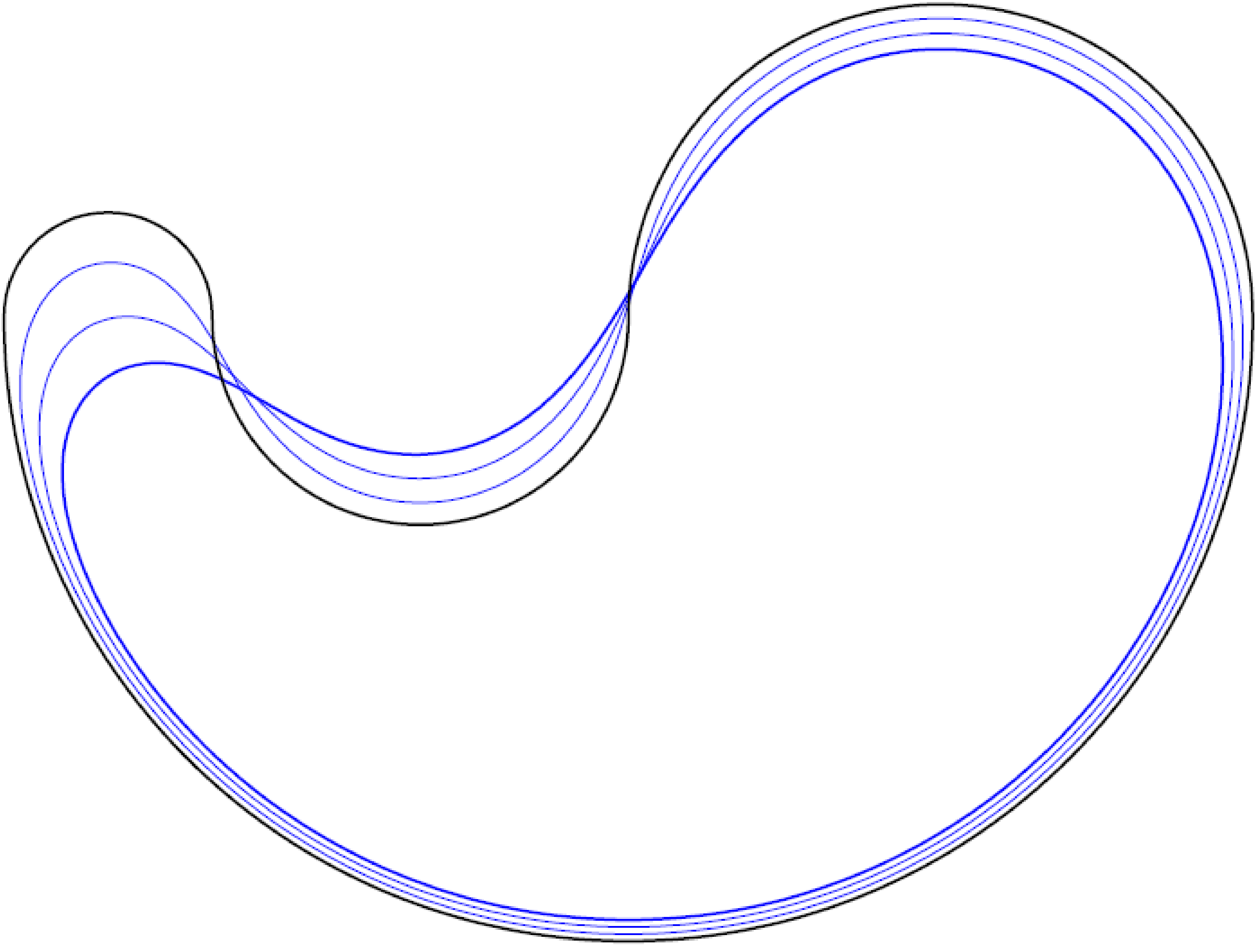


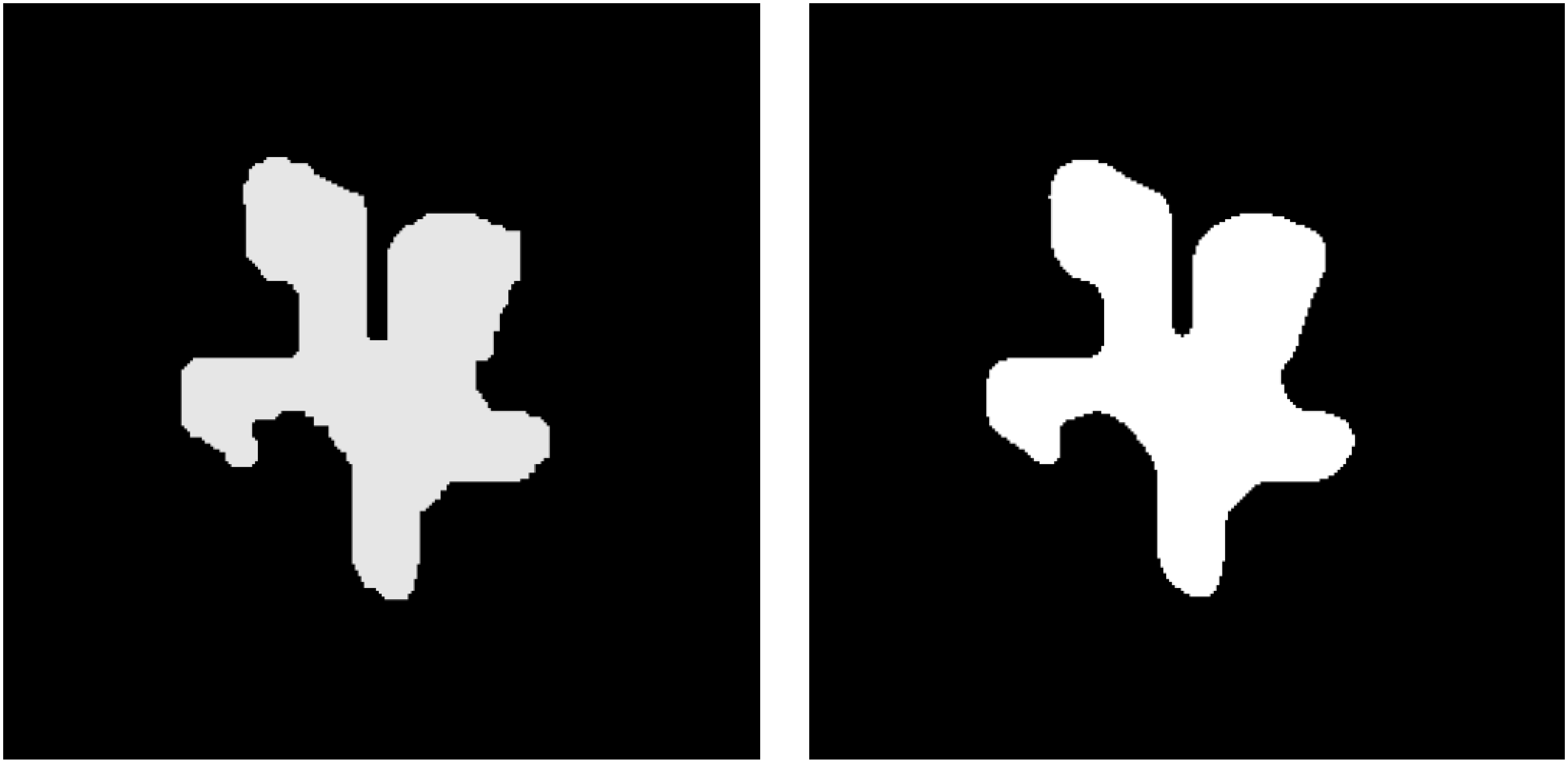


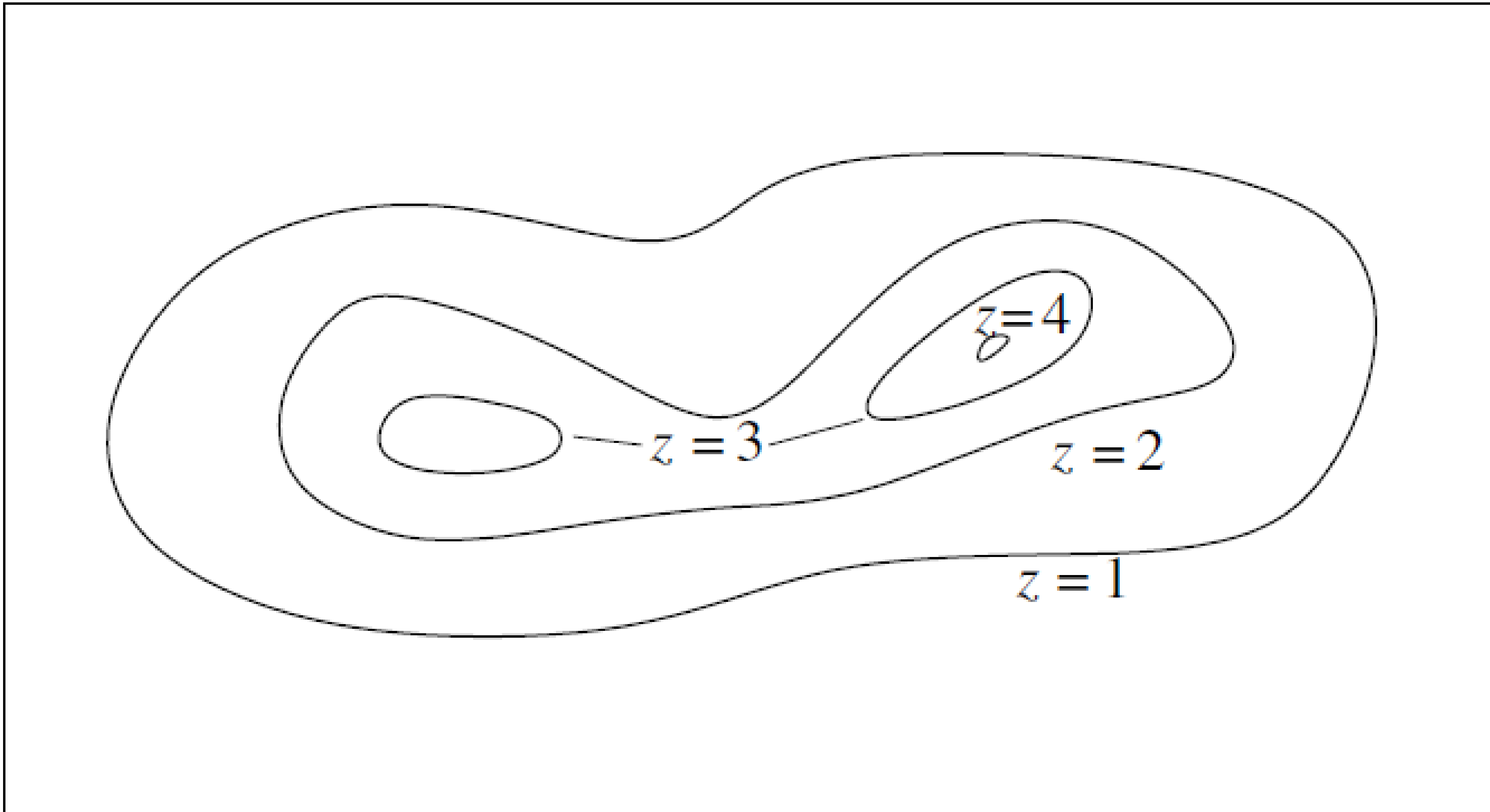




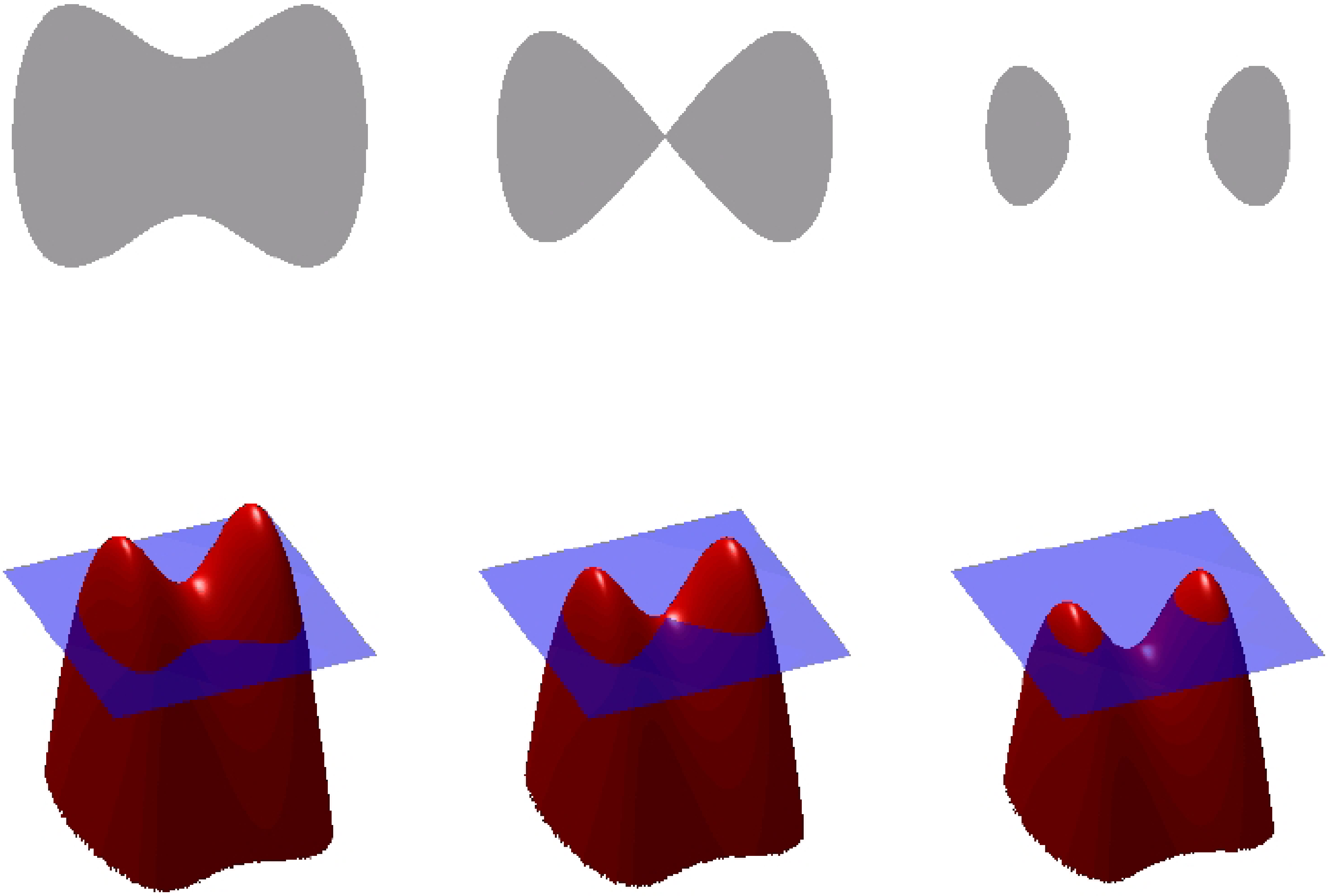


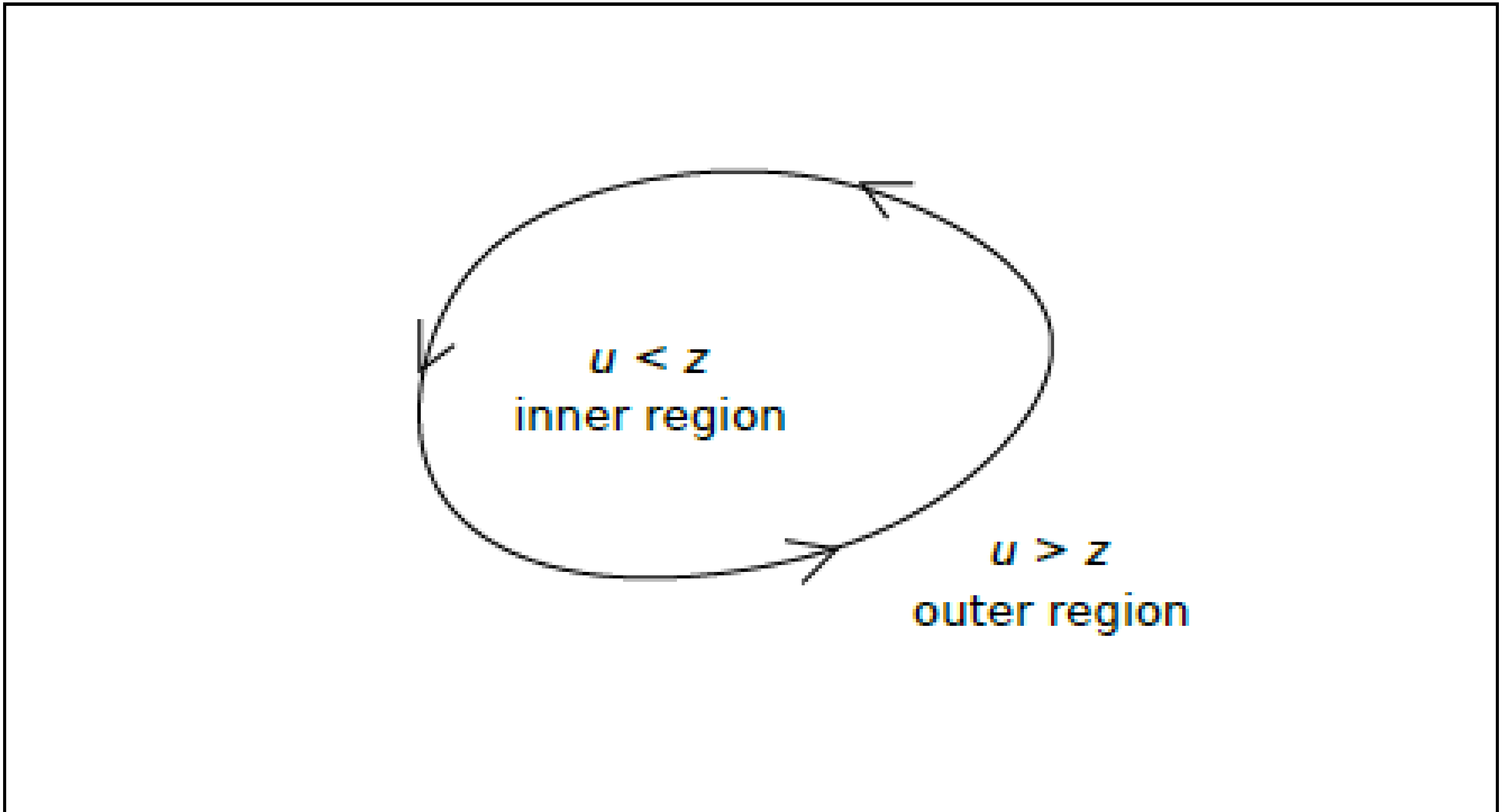


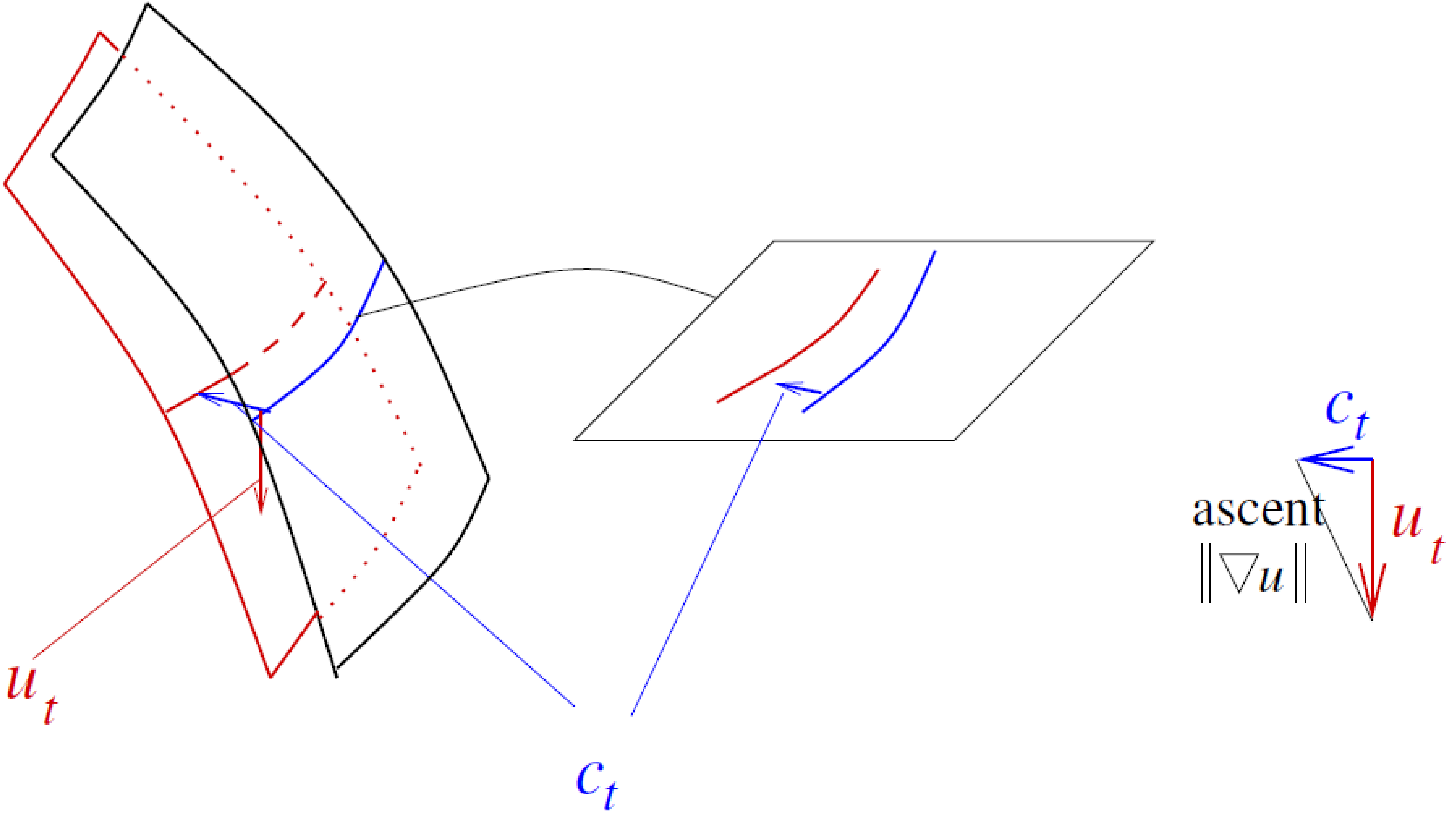




z = 4
z = 3
z = 2
z = 1




$u < z$
inner region
$u > z$
outer region



u_t
c_t
c_t
ascent
$\|\nabla u\|$
u_t