Lecture 2

- Manifolds
- Curves in $\mathbb{R}^d$
- Arc-Length
- Reparametrisation
- Curves in the Plane, Curvature
- Arc-Length with a Riemmanian Metric

Maps Between Manifolds

- consider manifolds M and N, and a mapping $F : M \to N$.
- F is a differential mapping between M and N if it is a differentiable function when restricted to charts
- If ϕ is a chart for M and ψ is a chart for N, then $\phi^{-1} \circ F \circ \psi$ is differentiable.
- F induces a linear transformation between tangent spaces

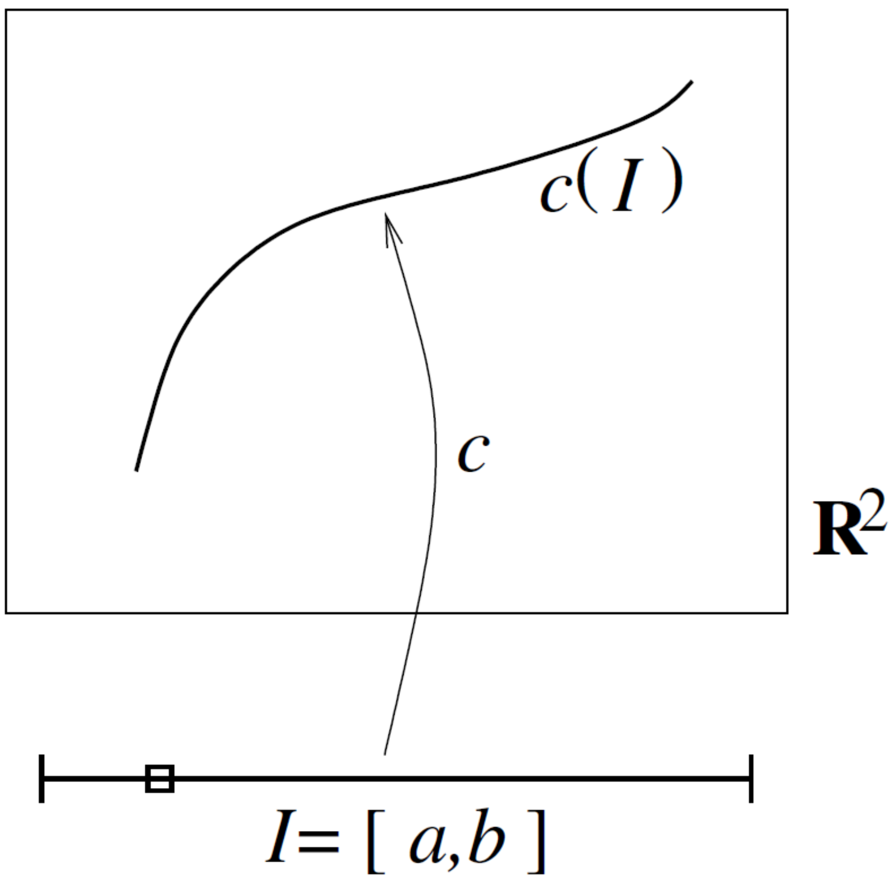
Related Concepts

- Manifold with boundary: similar to a manifold but maps neighbourhoods of certain (boundary) points to patches of the half-space $\mathbb{R}^{d-1} \times [0, \infty[$.
- Submanifold: If $M \subset N$ for manifolds M, N, and charts of M are restrictions of charts of N, then M is submanifold of N
- 1-D submanifolds of a manifold are curves
- 2-D submanifolds are surfaces

Curves in $\mathbb{R}^d$

- Curve in $\mathbb{R}^d$: differentiable function $c : I \to \mathbb{R}^d, I \subseteq \mathbb{R}$ interval
- The set $c(I) := \{c(p) : p \in I\}$ is the image or graph of c.

Remark: curves with identical graphs but different parametrisations are different



Related Definitions

- Regular curve: $c_p(p) \neq 0$ for all $p \in I$
- Closed curve: $I = [a, b]$, $c(a) = c(b)$, $c'_+(a) = c'_-(b)$
- Simple curve: for no $p1, p2 \in I$ with $p1 < p2$ the equality $c(p1) = c(p2)$ holds, except if c is a closed curve, $I = [a, b]$, and $p1 = a, p2 = b$

We assume always that curves are sufficiently often differentiable.

- k-regular curve: c differentiable k times, first k derivatives linearly independent in I

Remark: $c_p(p)$ is tangential vector for c in $x = c(p)$ (or, laxly speaking, in p), and $T_{c(p)}c \equiv \mathbb{R}$.

Length of a Curve

- $c : I \to \mathbb{R}^d$ curve parametrised with $I = [a, b]$
- Length of c:

$$L(c) := \int_a^b ||c_p(p)|| \, dp$$

where

$$||c_p(p)|| = \sqrt{\left(\frac{dx_1}{dp}\right)^2 + ... + \left(\frac{dx_d}{dp}\right)^2}$$

Reparametrisation

- transforms a curve into another curve with the same graph
- $c : I \to \mathbb{R}^d$ curve
- $\tilde{I} \subset \mathbb{R}$ interval
- Reparametrisation: a differentiable mapping $\phi : \tilde{I} \to I$, invertible with differentiable inverse
- Reparametrised curve: $\tilde{c} := c \circ \phi : \tilde{I} \to \mathbb{R}^d$
- orientation-preserving if $\phi'(\tilde{p}) > 0$ for all $\tilde{p} \in \tilde{I}$

Arc-Length Parametrisation

- curve $c : I \to \mathbb{R}^d$ is given in arc-length parametrisation if $I = [0, L]$, $L := L(c)$, and

$$||c_s(s)|| = 1 \quad \text{for all } s \in I$$

- For an arbitrarily parametrised regular curve $c : [a, b] \to \mathbb{R}^d$ the arc-length parametrisation is obtained by the transformation $\phi : s \to p$,

$$\frac{ds}{dp} = ||c_p(p)||, \qquad \left(s(p) = \int_a^p ||c_r(r)||\, dr \right).$$

Remarks:

- A curve in arc-length parametrisation is regular.
- We will use the parameter s (instead of p) for curves in arc-length parametrisation.

Curvature

- Let $c : I \to \mathbb{R}^2$ be arc-length parametrised.
- Then
$$< c_s(s), c_s(s) >= 1$$
- and by differentiation
$$< c_s(s), c_{ss}(s) >= 0$$
i.e.
$$c_{ss}(s) \perp c_s(s).$$

Remark: The arc-length parametrisation is essential for this orthogonality!

Curvature

- Let $\vec{t}(s), \vec{n}(s)$ be unit vectors tangential and normal to c at $c(s)$, resp., and $(\vec{t}, \vec{n})$ positively oriented
- Then
$$c_s(s) = \vec{t}(s) \quad c_{ss}(s) = \kappa(s)\vec{n}(s)$$
with a uniquely determined function $\kappa(s)$
- $\kappa(s)$ is called curvature of c at $c(s)$

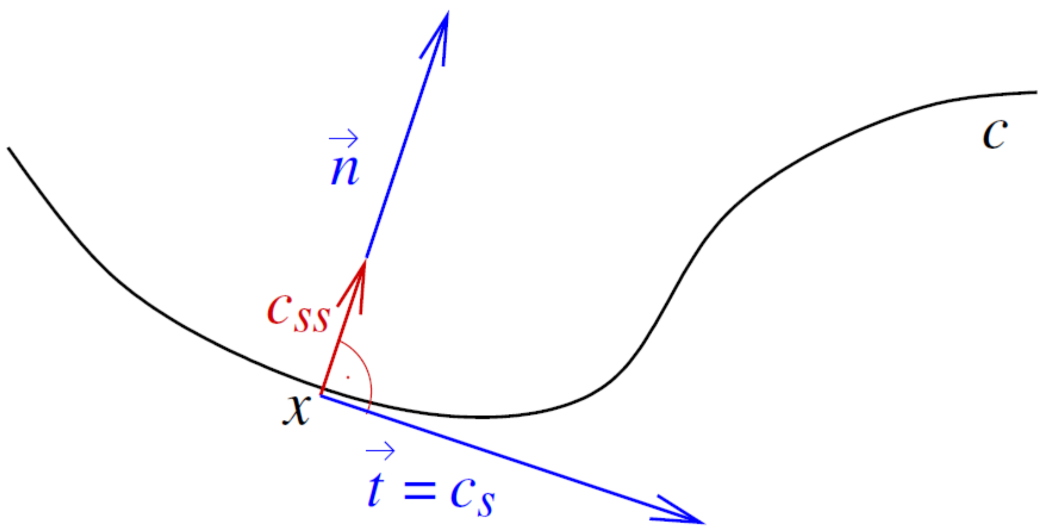


Figure: Curve c with tangent and normal vectors, first and second derivatives at point $x = c(s)$.

Curvature

- If $\rho(s)$ is the radius of the osculating circle for c at $c(s)$ (i.e. the circle that best approximates the curve in $c(s)$), then $\rho(s) = \frac{1}{|\kappa(s)|}$

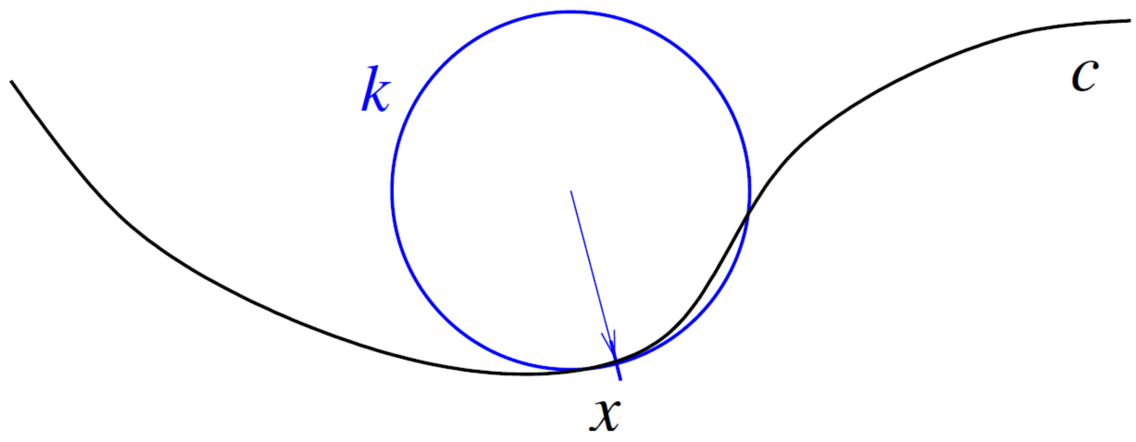


Figure: Curve c with osculating circle k at point $x = c(s)$

Curvature

The osculating circle at $c(p)$ is characterised by:

- passes through $c(p)$
- has a common tangent line at $c(p)$
- near $c(p)$ distance between curve and circle when following the normal direction of c decays rapidly

For an arc-parametrised curve the center of the the osculating circle Q is:

$$Q(s) = c(s) + \frac{c_{ss}(s)}{||c_{ss}(s)||^2}$$

For other parametrisations:

$$Q(p) = c(p) + \frac{1}{\kappa(p)||c_p(p)||}(-x_2'(p), x_1'(p)), \quad \text{with} \quad \kappa(p) = \frac{x_1'(p)x_2''(p) - x_1''(p)x_2'(p)}{(x_1'(p)^2 + x_2'(p)^2)^{\frac{3}{2}}}$$

Determination of Curves by Curvature

The function $\kappa(s)$ determines the curve $c(s)$ up to translations (initial point $(x_1(0), x_2(0))$) and rotations (initial direction $\phi(0)$):

$$\phi(s) = \phi(0) + \int_0^s \kappa(\sigma)\, d\sigma$$

$$x_1(s) = x_1(0) + \int_0^s \cos\phi(\sigma)\, d\sigma \quad x_2(s) = x_2(0) + \int_0^s \sin\phi(\sigma)\, d\sigma$$

Special Properties of Planar Curves

The following hold for curves in the Euclidean plane.

- The only planar curves of constant curvature are straight lines and circles.
- The total curvature $\oint_c \kappa(s)\,ds$ of a closed planar curve is a multiple of 2π.
  - For a simple curve, it is $\pm 2\pi$.
  - The integer quantity $\frac{1}{2\pi}\oint_c \kappa(s)\,ds$ is called rotation number.

Riemmanian Metrics

- The tangent spaces of a manifold can be complemented with a Riemmanian metric: a positive bilinear form acting on the d-dimensional vector space corresponding to the tangent space at each point of the manifold.

$$g_{\phi(p)} : T_{\phi(p)}M \times T_{\phi(p)}M \to \mathbb{R}$$

- The bilinear form should variate smoothly over the manifold.
- In the case of a curve it is nothing else than a rescaling.

Length of a Curve with a Riemannian Metric

- $M = \mathbb{R}^d$ Riemannian manifold with metric g
- $c : I \to M$ curve parametrised with $I = [a, b]$
- Length of c :

$$L_g(c) := \int_a^b ||c_p(p)||_g \, dp$$

- $c_p(p)$ is a tangent vector in $T_{c(p)}(c)$, and the norm $|| \cdot ||_g$ depends on the metric g as follows

$$||c_p(p)||_g = \sqrt{g_{c(p)} \left(c_p(p), c_p(c) \right)}$$

- In Euclidean metric:

$$L(c) = \sqrt{\left(\frac{dx_1}{dp} \right)^2 + ... + \left(\frac{dx_d}{dp} \right)^2}$$

Arc-Length Parametrisation with a Riemannian Metric

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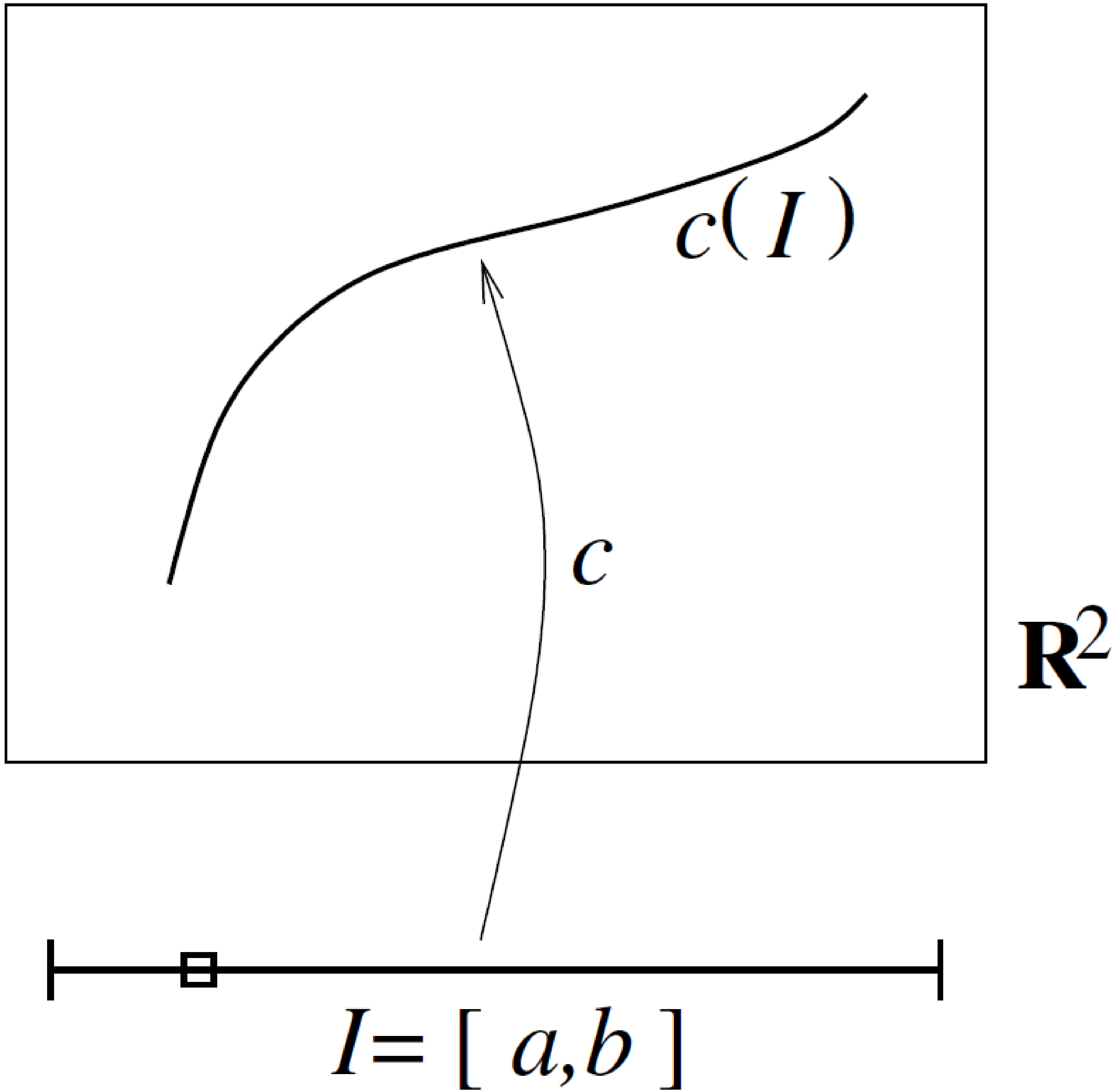
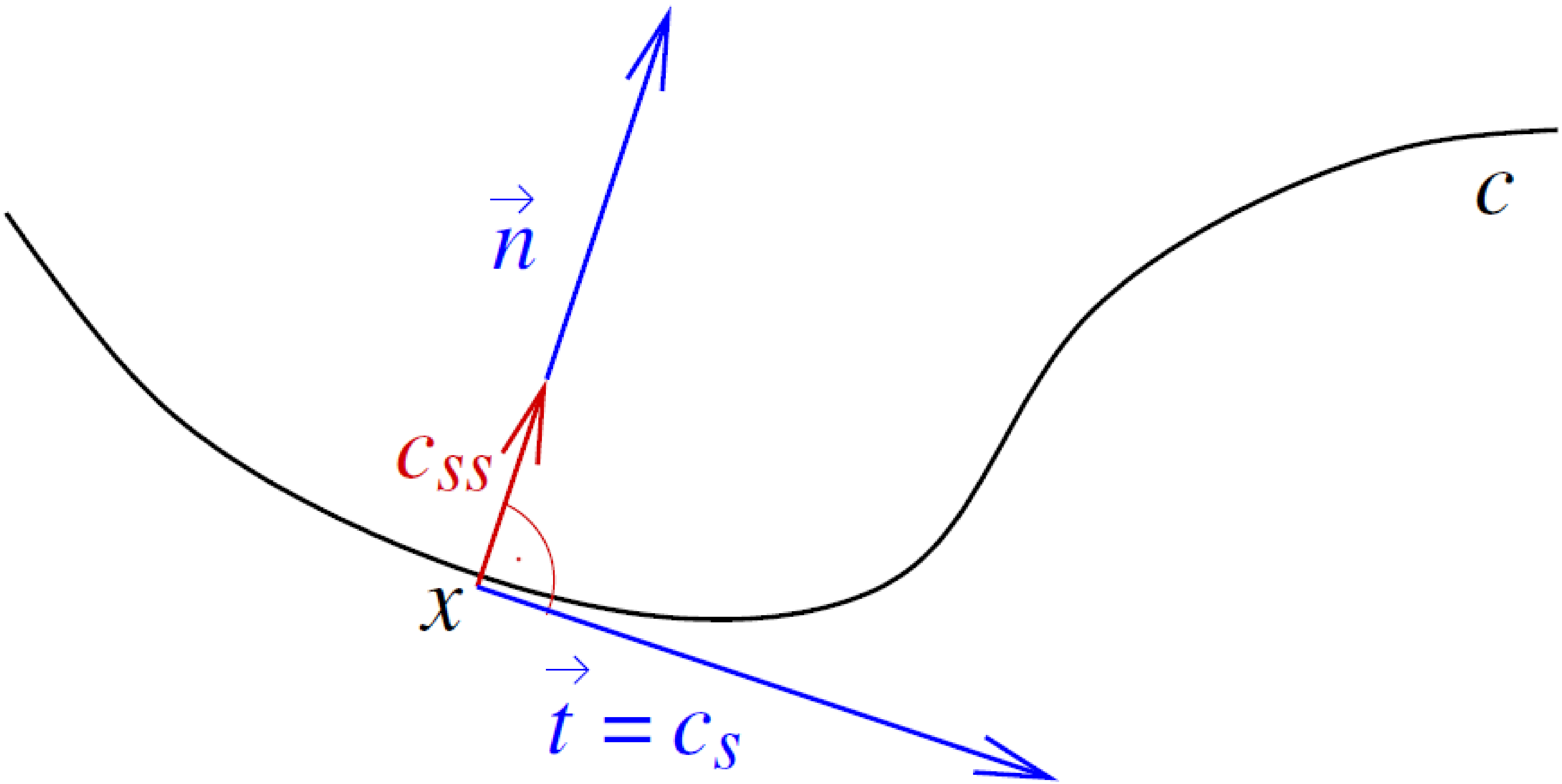
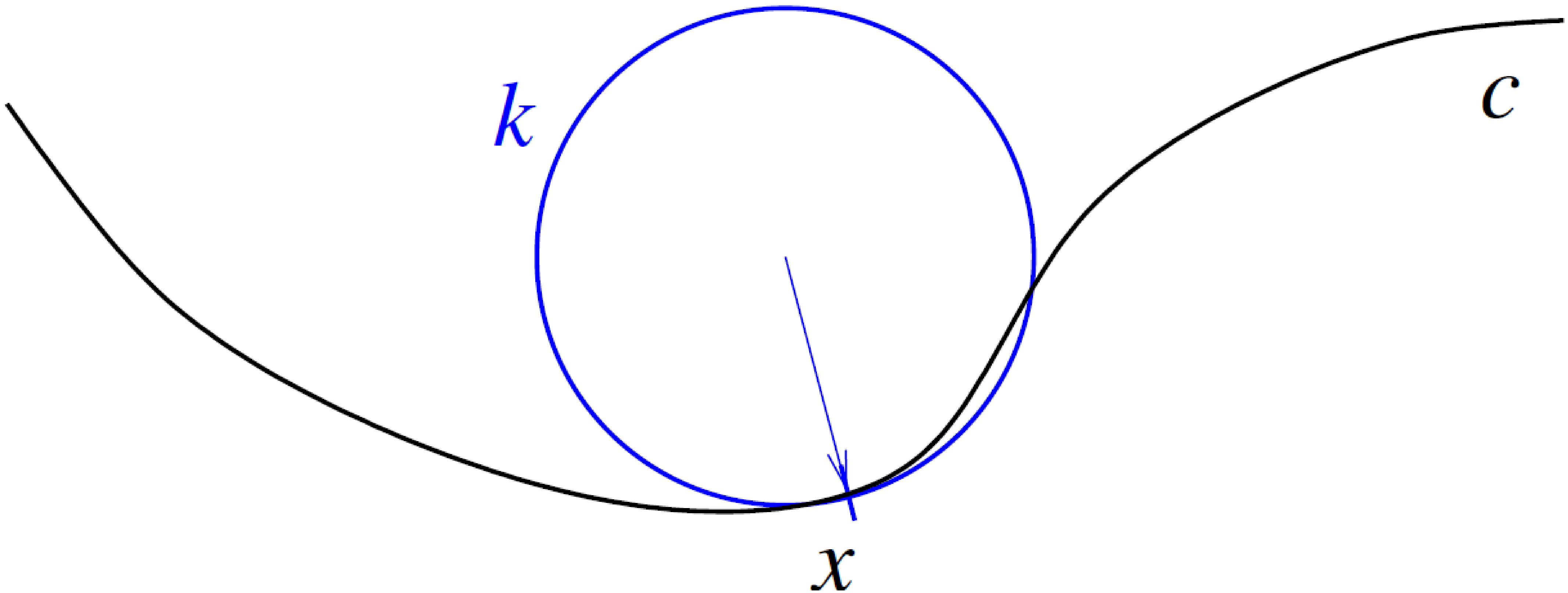
Riemannian Manifold as Metric Space

- $M = \mathbb{R}^d$ Riemannian manifold with metric g
- Let two points $x, y \in M$ be given
- Define a distance function $d(x,y) := \min\{L_g(c) | \, c : [0,1] \to M \,\text{curve}, c(0) = x, c(1) = y\}$ i.e. the shortest length of a curve on M joining x and y
- (M, d) is a metric space
- Length-minimising curves as in the definition of d but in arc-length parametrisation are geodesics on M
- Geodesics play an outstanding role in the structure of the Riemannian manifold. More about geodesics on surfaces in a later lecture



$c(I)$
c
$\mathbf{R}^2$
$I = [a, b]$


$\vec{n}$
c
c_{ss}
x
$\vec{t} = c_s$



k
c
x