Differential Geometric Aspects of Image Processing

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http://www.mia.uni-saarland.de/cardenas/index.shtml

Winter Term 2019
http://www.mia.uni-saarland.de/Teaching/dgip19.shtml

What is this all about? (1)

- specialised lecture in image processing
- related to, but not dependent on
  - Image Processing and Computer Vision (offered regularly in summer terms)
  - Differential Equations in Image Processing and Computer Vision (offered this semester by Prof. Dr. Joachim Weickert)

and further specialised courses.

What is this all about? (2)

- applications of differential geometric ideas in image processing
- concepts of differential geometry often allow a particularly elegant formulation and derivation of image processing methods
- variational and PDE (partial differential equation) formulations play an important role
- wide range of applications:
  - image denoising
  - image enhancement
  - detection of structures (e.g. shapes/contours)
  - processing of shape information
  - deblurring of images
- necessary mathematical instruments will be provided, focus is on application

The course does not follow a specific book. The following books cover many of its topics:

- F. Cao, Geometric Curve Evolution and Image Processing. Lecture Notes in Mathematics, vol. 1805, Springer, Berlin 2003
- R. Kimmel, Numerical Geometry of Images. Springer, Berlin 2004
- S. Osher, N. Paragios, eds., Geometric Level Set Methods in Imaging, Vision and Graphics. Springer, Berlin 2003
- G. Sapiro, Geometric Partial Differential Equations and Image Analysis. Cambridge University Press 2001.

Further references will be given where appropriate.

General Schedule

- Workload: 4 hours per week, with exercises (1 hour approx.), 6 credit points
- Lectures on Tuesdays, 16–18, and Thursdays, 14–16.
- Building E1.3, Lecture Hall 003.
- First tutorial: October 24, 2019.
- Registration: you can and should register until next Monday (send a mail to cardenas@mia.uni-saarland.de with your name and your course of studies)
- Slides will be available under http://www.mia.uni-saarland.de/Teaching/dgip19.shtml for password-protected download.

Overview of Topics

- Basic differential geometric concepts
- Curves in plane, curve evolution, shapes evolution
- Level sets, level set formulations of PDE-based image filters
- Curves and surfaces in space
- Image filtering on surfaces
- Image domains with non-Euclidean metrics, and corresponding filters
- Variational problems, Euler-Lagrange equations and gradient descents
- Surface evolution
- Filtering of surfaces
- The Beltrami framework
- Geodesic active contours and regions, and related methods

Continuous-Scale Images

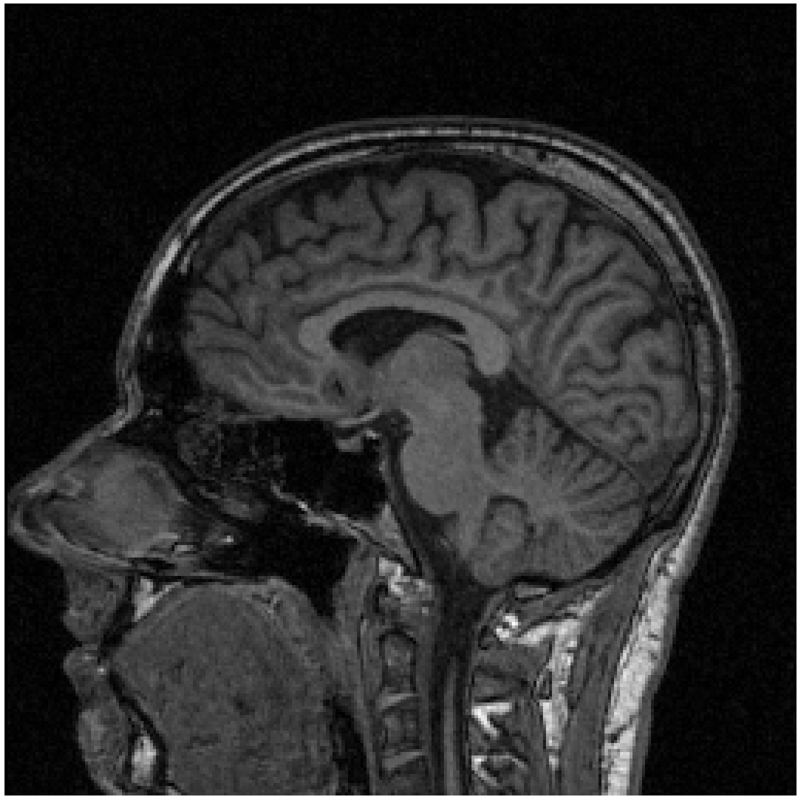
In most of this course, we think of images as functions between manifolds.
For now consider two important examples:

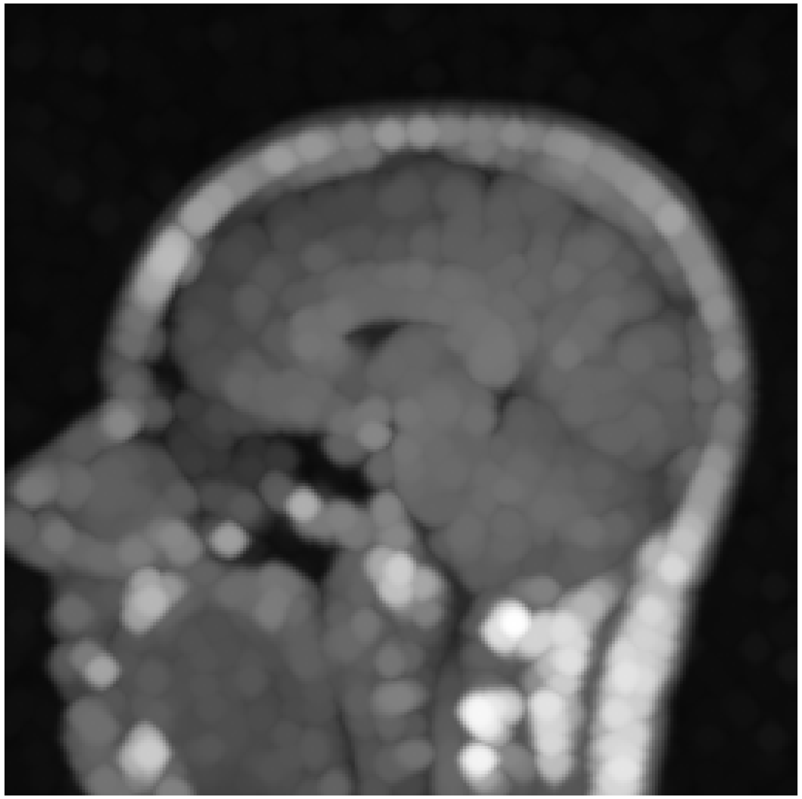
- grey-value image (one possibility): from a bounded domain (manifold with boundary) in $\mathbb{R}^2$ to $\mathbb{R}$
- colour image: from bounded domain in $\mathbb{R}^2$ to $\mathbb{R}^3$
- examples with more complicated manifolds later in this course

Discrete Images

- In practice, both the domain and the range of images are discretised.
  - domain discretisation: sampling
  - range discretisation: quantisation
- only partial coverage of numerical algorithms for discrete images in this course

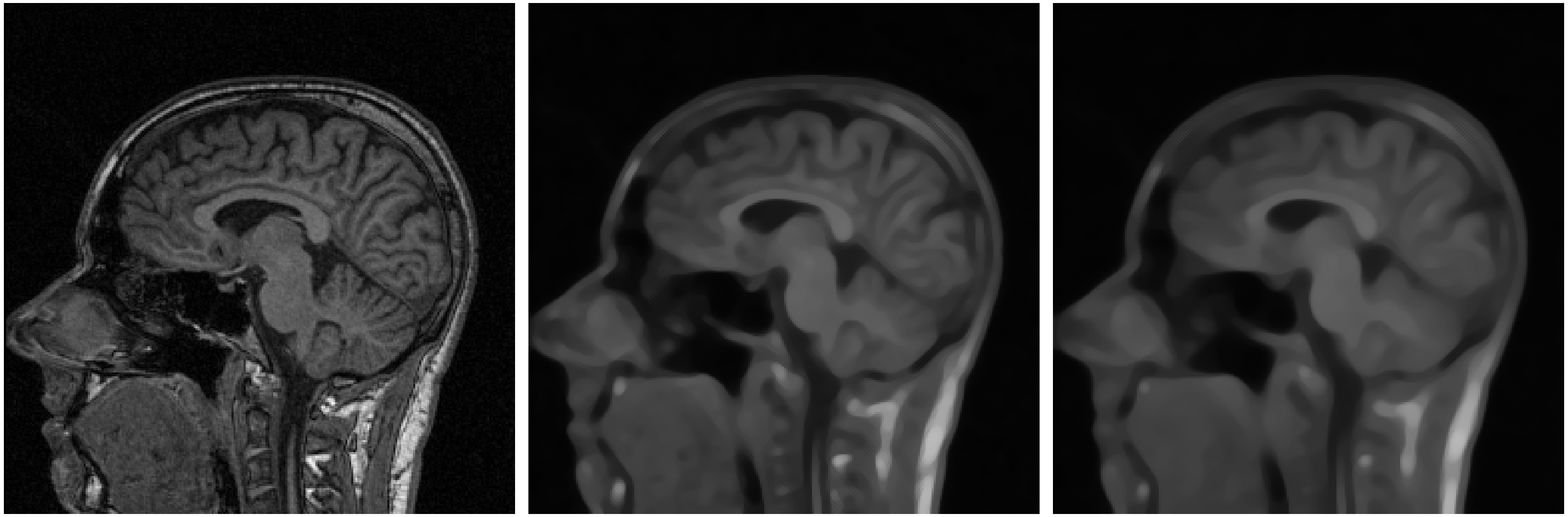
Dilation





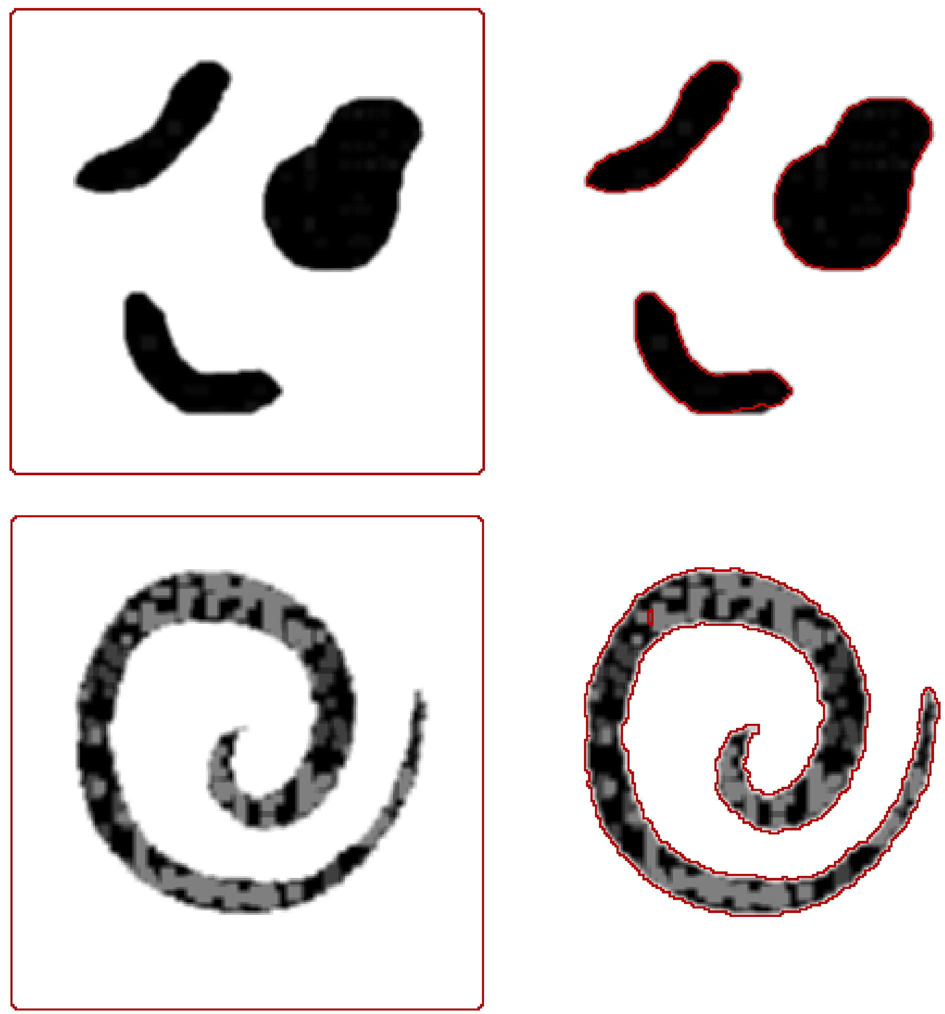
Left: MR image of a human head. **Right:** Processed by dilation, which can be described as a curve evolution process and acts on the shapes of objects.

Mean Curvature Motion



Left to right: MR image of a human head; processed by curvature motion with two different evolution times. Curvature motion is also a curve evolution process.

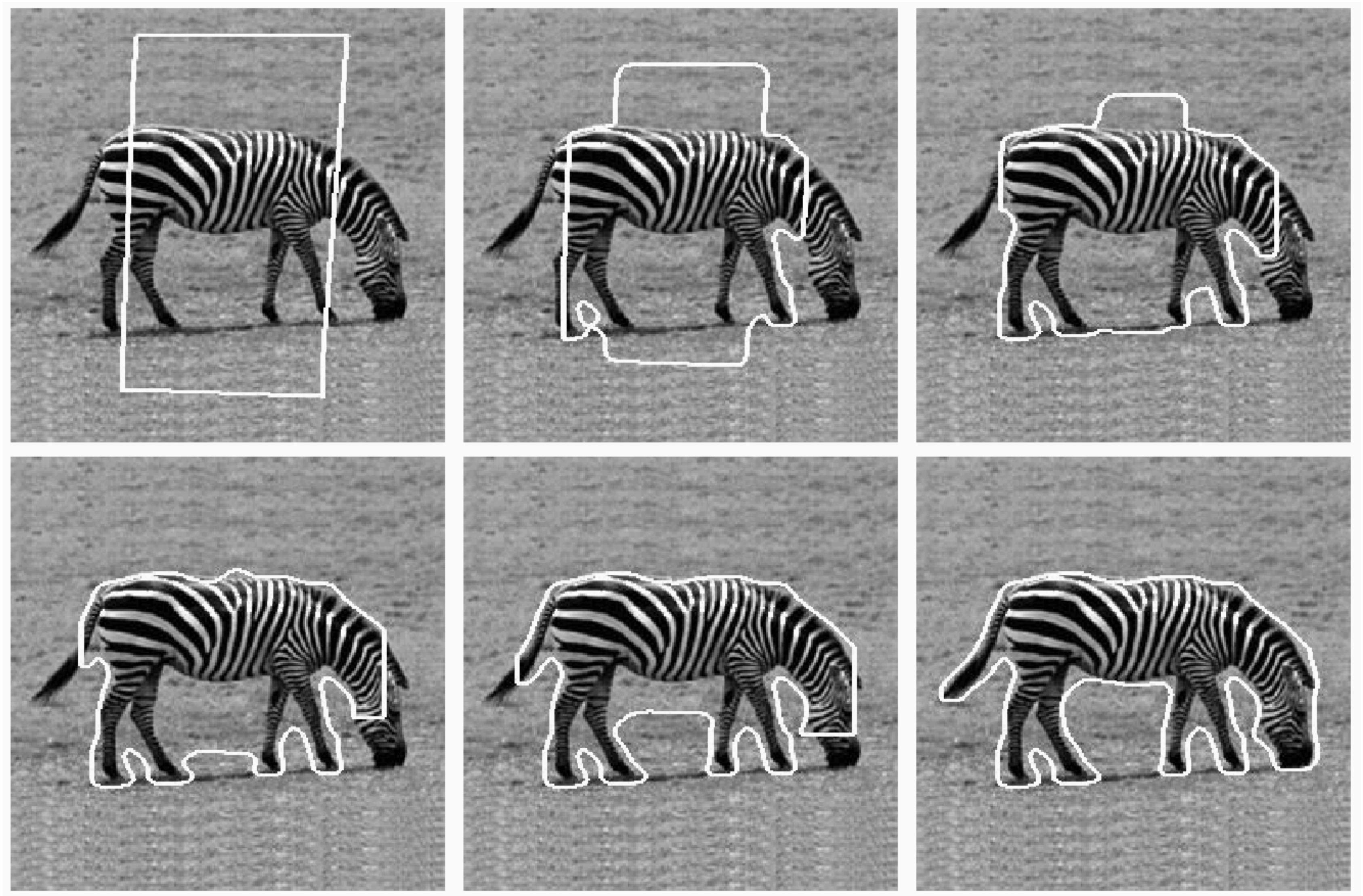
Geodesic Active Contours



Geodesic active contours (after Kichenassamy et al., 1996). **Top left, bottom left:** Original images with initial contours. **Right:** Results of geodesic active contour computation.

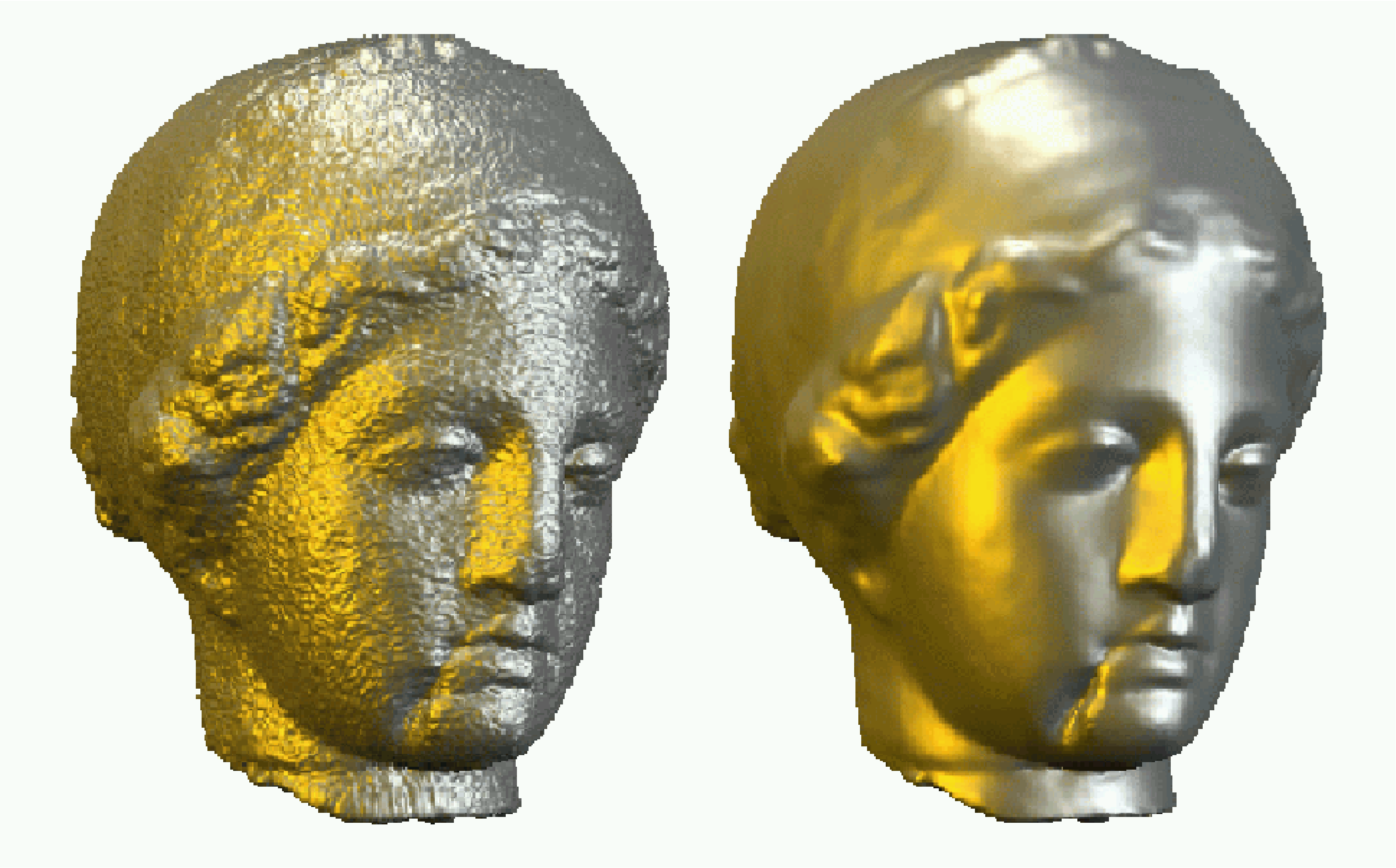
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Image Segmentation



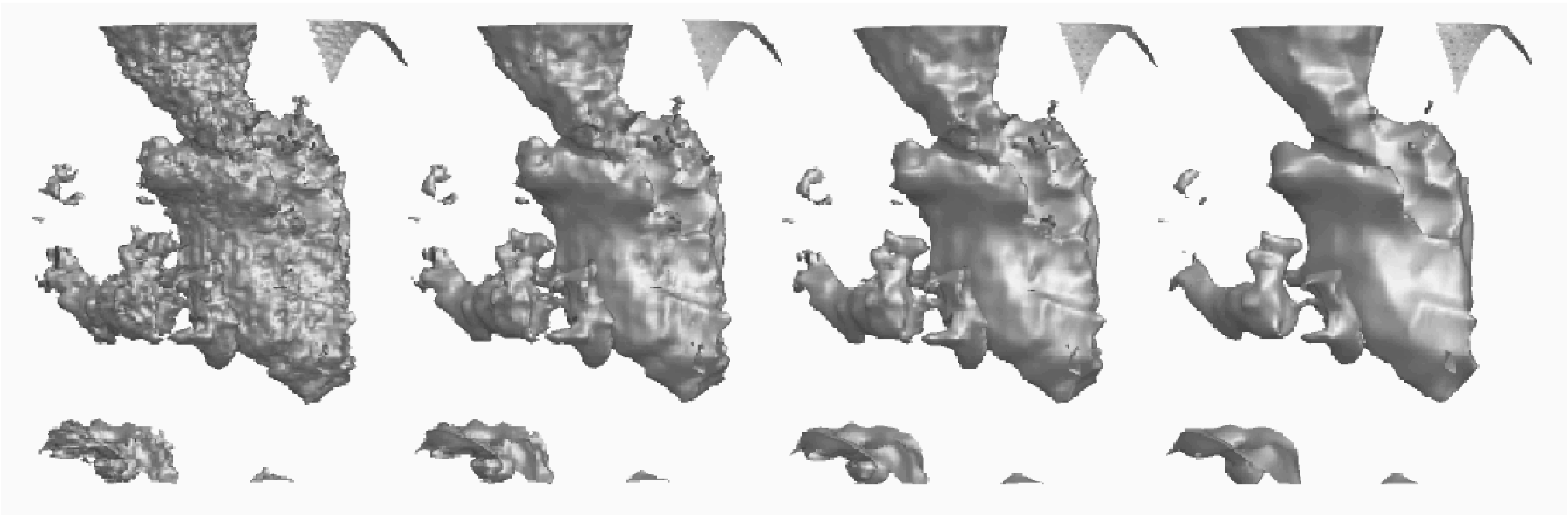
Object-background segmentation using a texture-controlled geodesic active region model. (N. Paragios, R. Deriche 2002)

Surface Smoothing



A venus sculpture surface smoothed by anisotropic geometric diffusion. (T. Preuer, M. Rumpf 2002)

Level Set Smoothing



Human heart ventricle extracted as a level set from an echocardiographic image, smoothed successively by anisotropic geometric diffusion. (T. Preuer, M. Rumpf 2002)

Texture Surface Smoothing

Smoothing of a laser-scanned surface with onscribed texture. **Left to right:** Original surface; contaminated with isotropic noise; the noisy surface smoothed by pure geometric diffusion; same with combined geometry and texture evolution (Clarenz, Diewald, Rumpf 2003).

Review of Basic Concepts

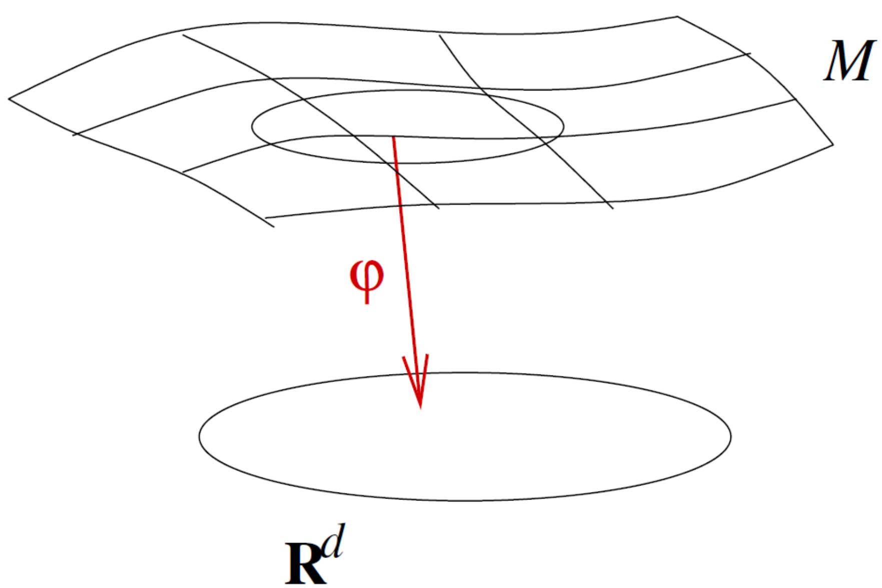
- Consider the Euclidean vector space $\mathbb{R}^d$.
  - If $\vec{v}, \vec{w} \in \mathbb{R}^d$ and $\vec{v} = (v_1, ..., v_d)^\top, \vec{w} = (w_1, ..., w_d)^\top$, their scalar product is $\langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^d v_i w_i$.
  - The Euclidean norm of $\vec{v} \in \mathbb{R}^d$ is given by $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \left(\sum_{i=1}^d v_i^2\right)^{1/2}$.
  - $\vec{v}, \vec{w} \in \mathbb{R}^d$ are called orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$. In that case we write $\vec{v} \perp \vec{w}$
  - $\vec{v_1}, \vec{v_2}, ..., \vec{v_m} \in \mathbb{R}^d$ are linear independent if for any $\lambda_1, \lambda_2, ..., \lambda_m \in \mathbb{R}$, $\sum_{i=1}^m \lambda_i \vec{v_i} = 0$ implies $\lambda_i = 0$ for all i
  - linear transformations, orthogonal matrices, ...
  - symmetric positive definite bilinear forms/matrices

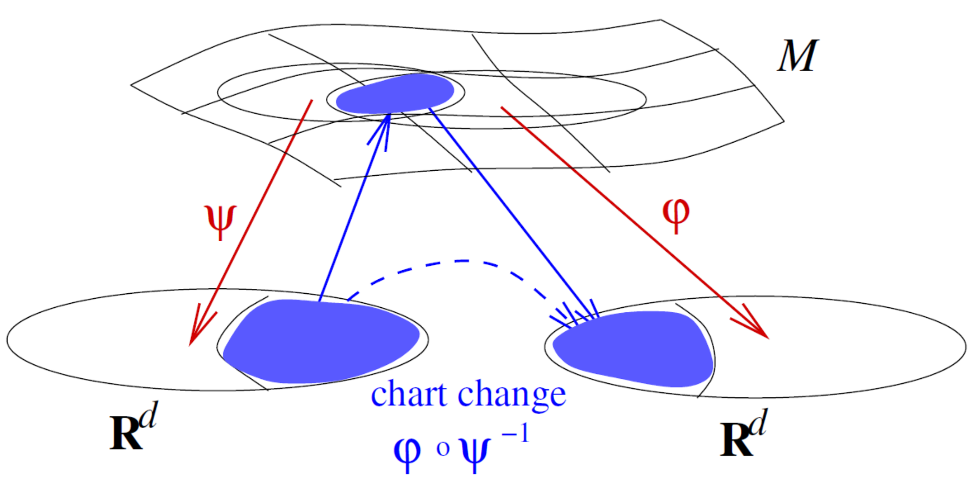
Review of Basic Concepts

- Consider a function $f : \Omega \subset \mathbb{R}^d \to \mathbb{R}^n$ with $f = (f_1, f_2, ..., f_n)^\top$.
  - f is smooth if all its components are smooth
  - if $f(x_1, ..., x_d) : \mathbb{R}^d \to \mathbb{R}$ is a differentiable function we will denote its partial derivatives $\frac{\partial f}{\partial x_i}$ also with f_{x_i}
  - for $n = 1$, $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_d})^\top = (f_{x_1}, f_{x_2}, ..., f_{x_d})^\top$

Intuitive definition of a general manifold

- any manifold M can be locally parameterised via continuous mappings (charts) to open sets of $\mathbb{R}^d$ and d corresponds to its dimension
- for now we assume that $M \subset \mathbb{R}^m$
- charts should be invertible
- different charts of the same manifold should be coherent (chart changes are smooth functions)
- a set of coherent charts which cover all M is called atlas





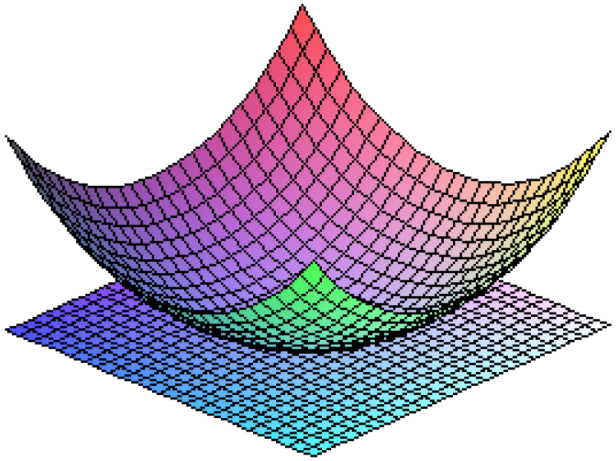
Simple Examples of Manifold

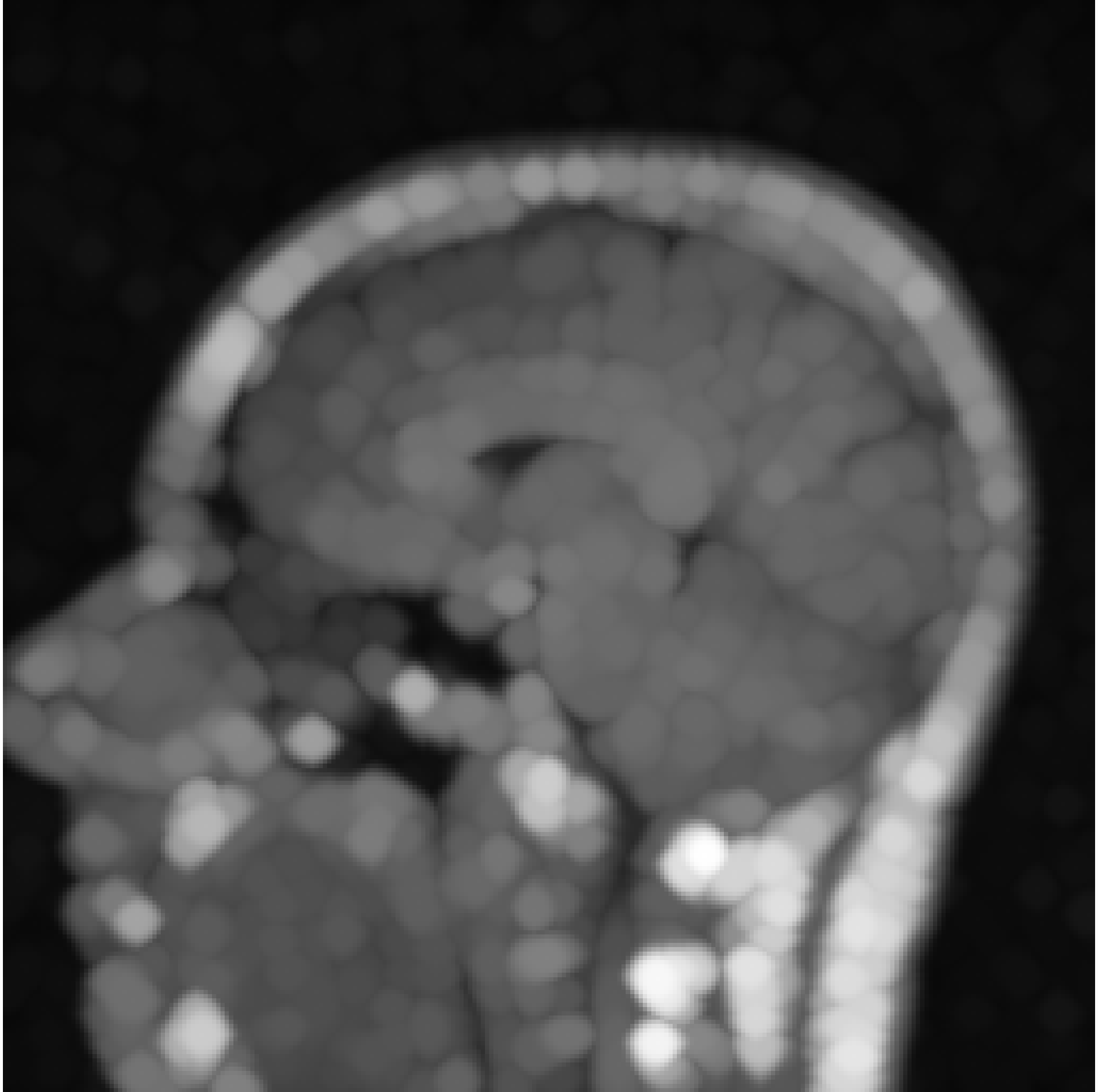
- $\mathbb{R}^d$
- smooth simple curve
- circle
- sphere
- torus
- etc...

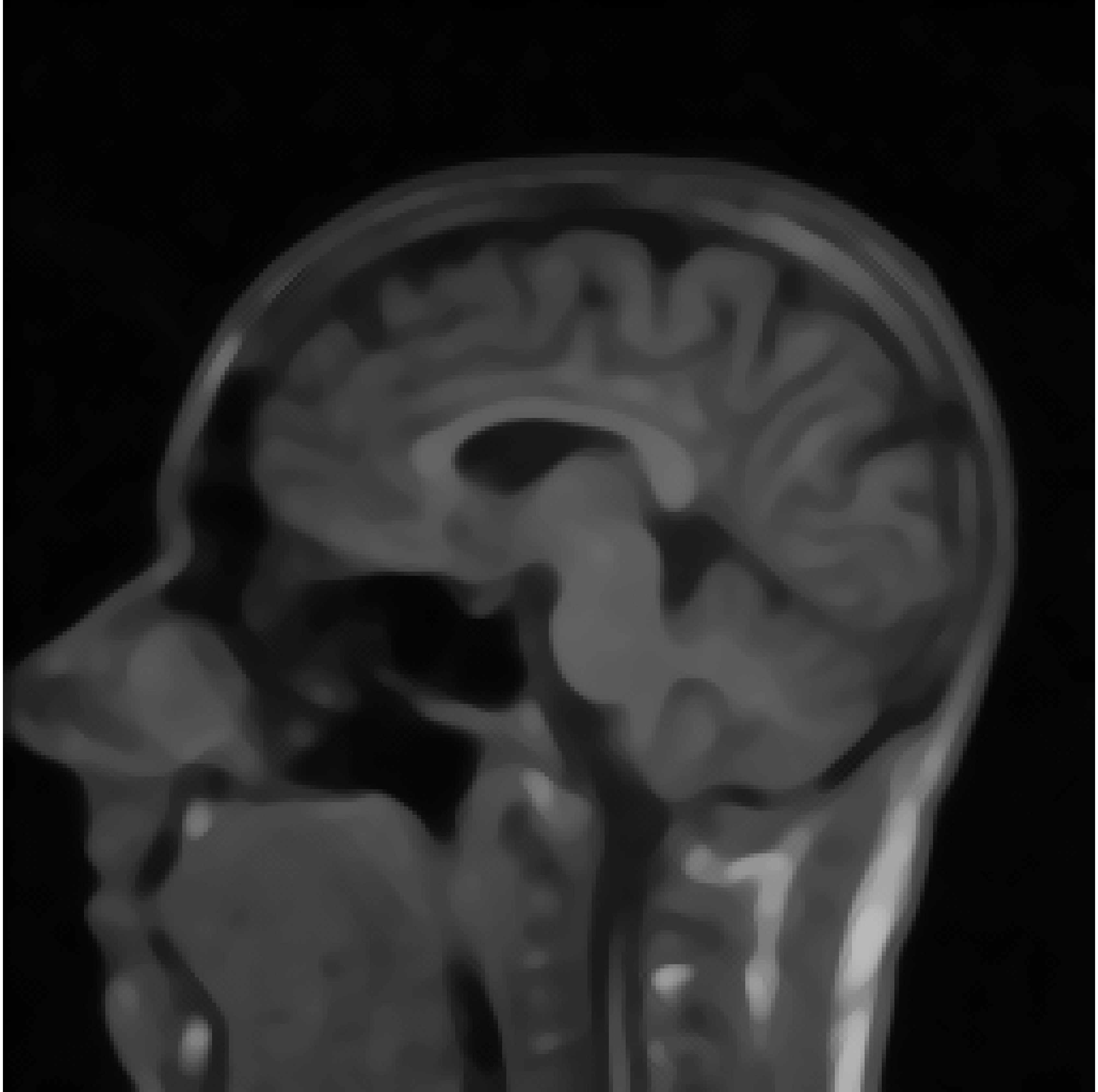
Tangent Spaces

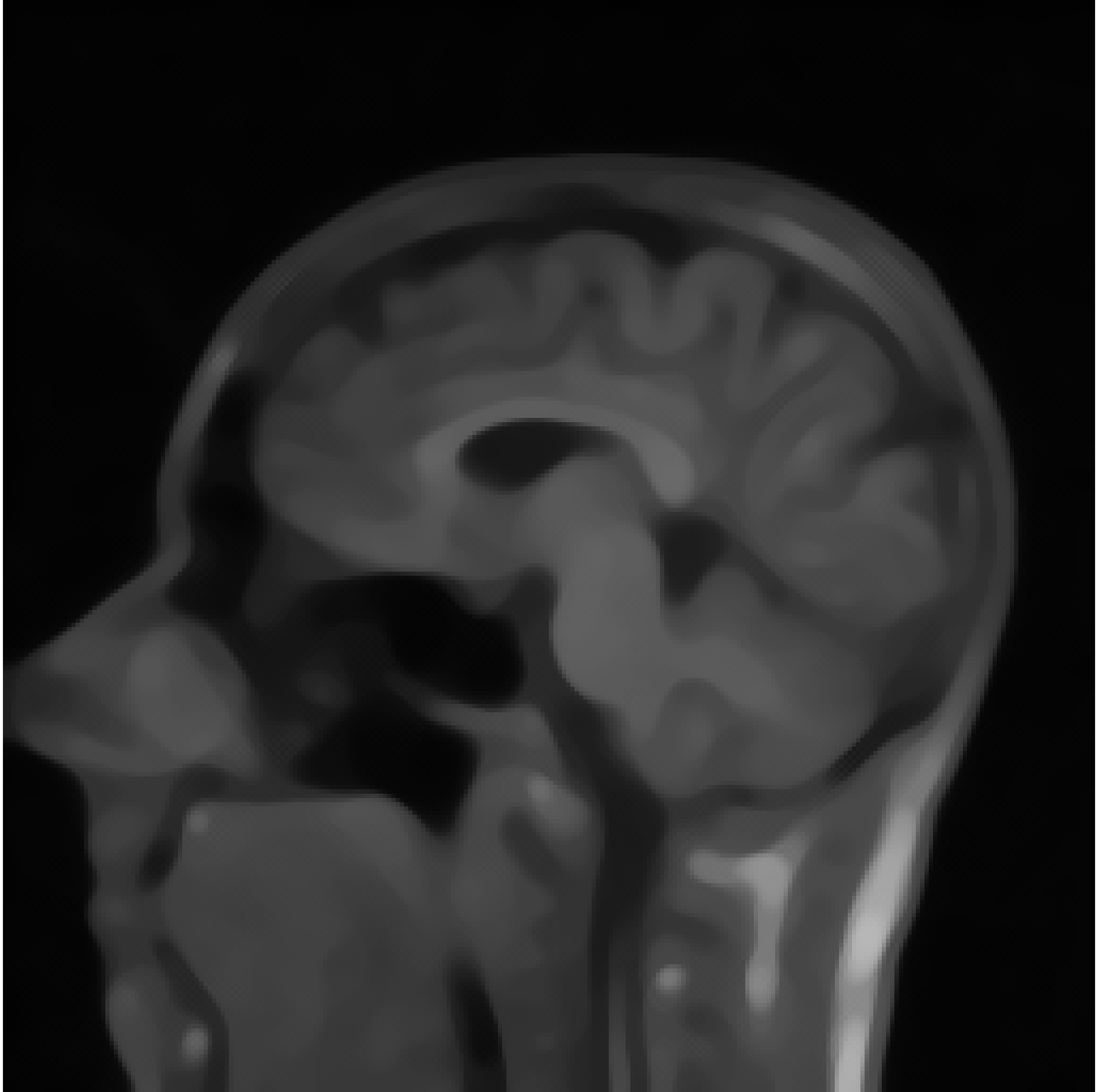
- Let $c : I \to \mathbb{R}^d$ be a smooth function (chart of a curve/1-dimensional manifold) defined on an interval $I \in \mathbb{R}$.
- The vector $\frac{dc}{dt}(p)$ is a vector with same direction as the tangent line touching the curve given by the graph of c at point $c(p)$.
- This tangent line is the tangent space of c at $c(p)$, $T_{c(p)}(c)$. We can identify it with $\mathbb{R}$.

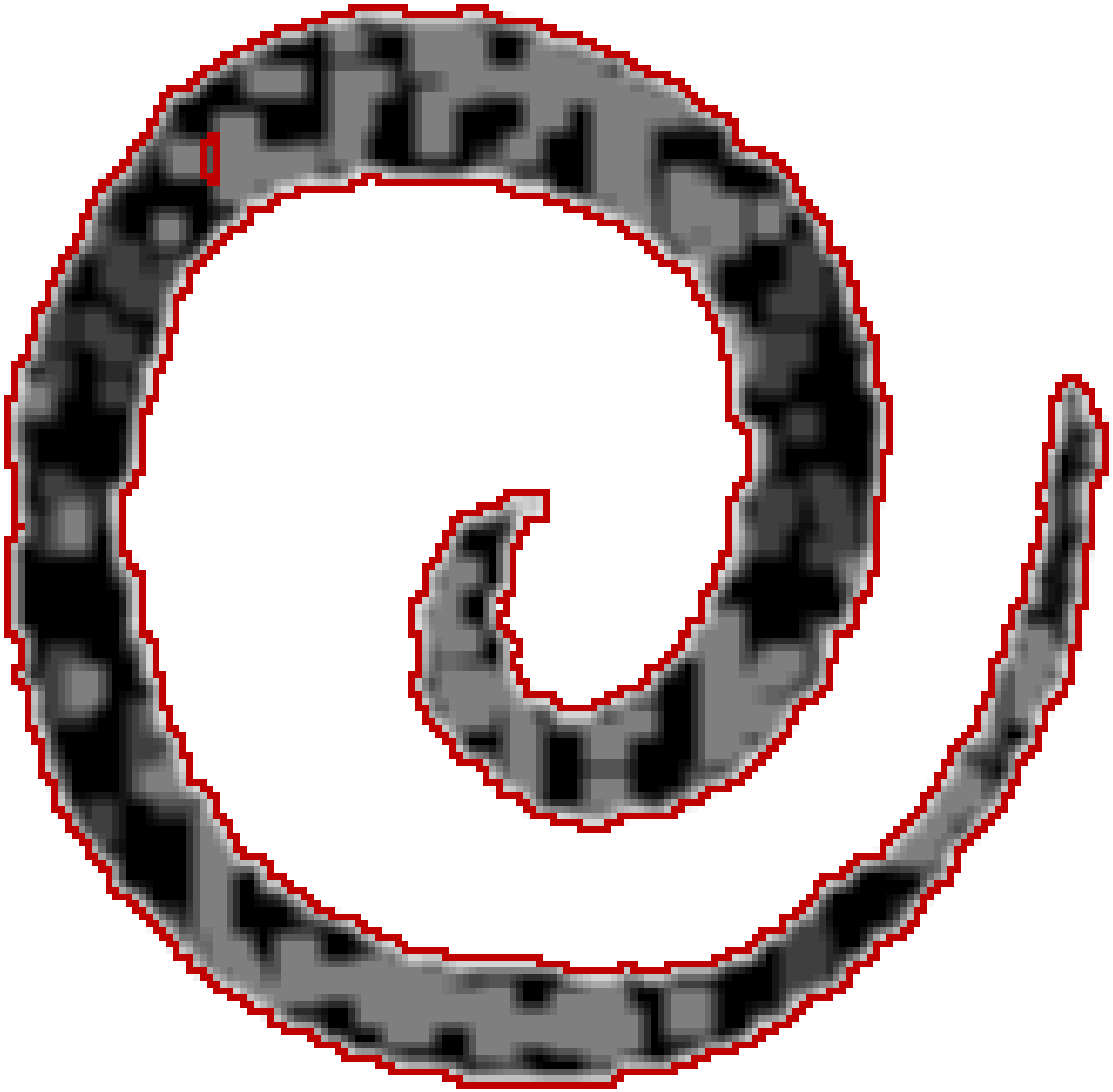
The same concept can be generalised for a generic manifold, think of the tangent plane of a surface.
The tangent space of a d-dimensional manifold can be identified with $\mathbb{R}^d$.

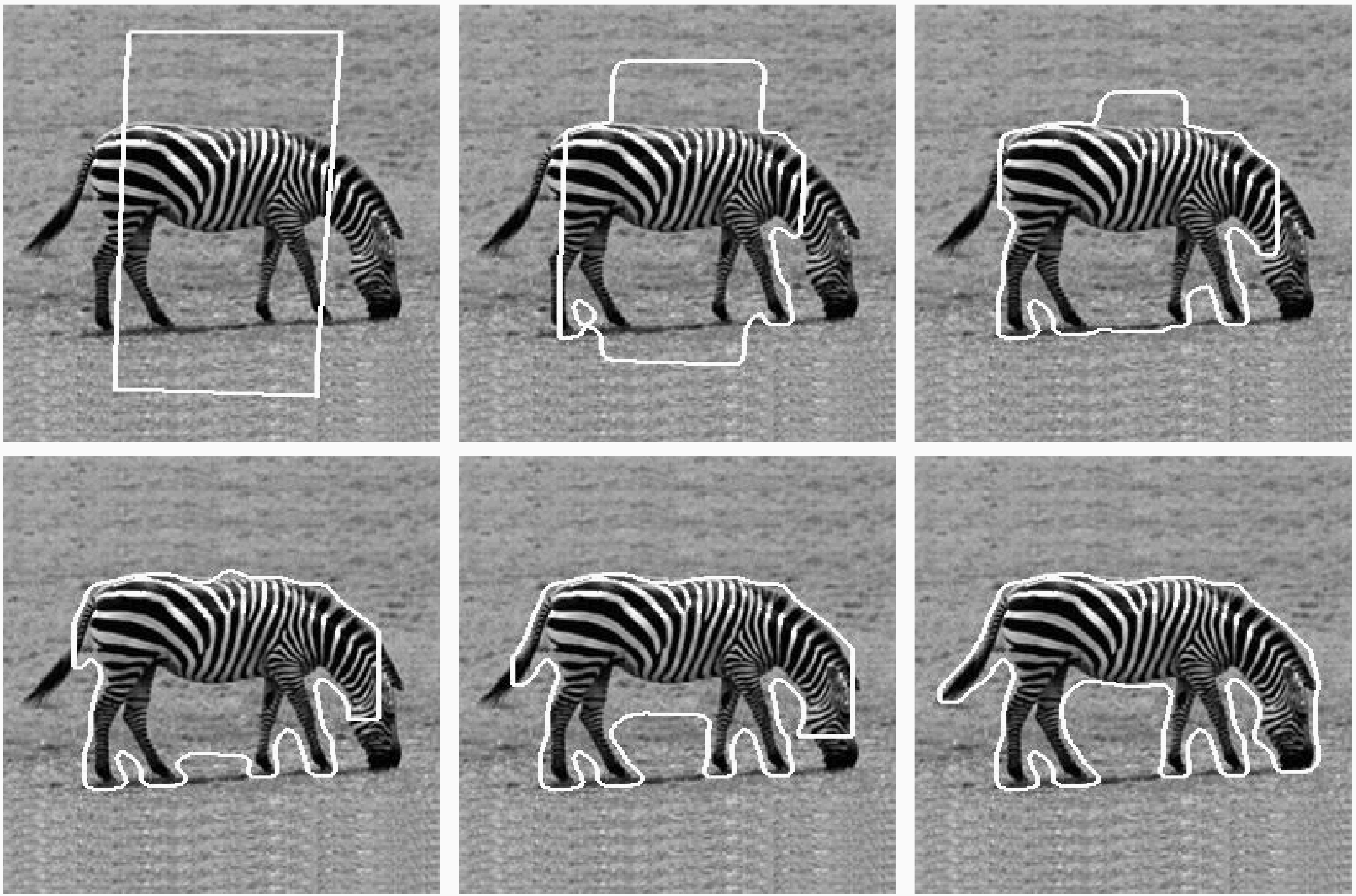


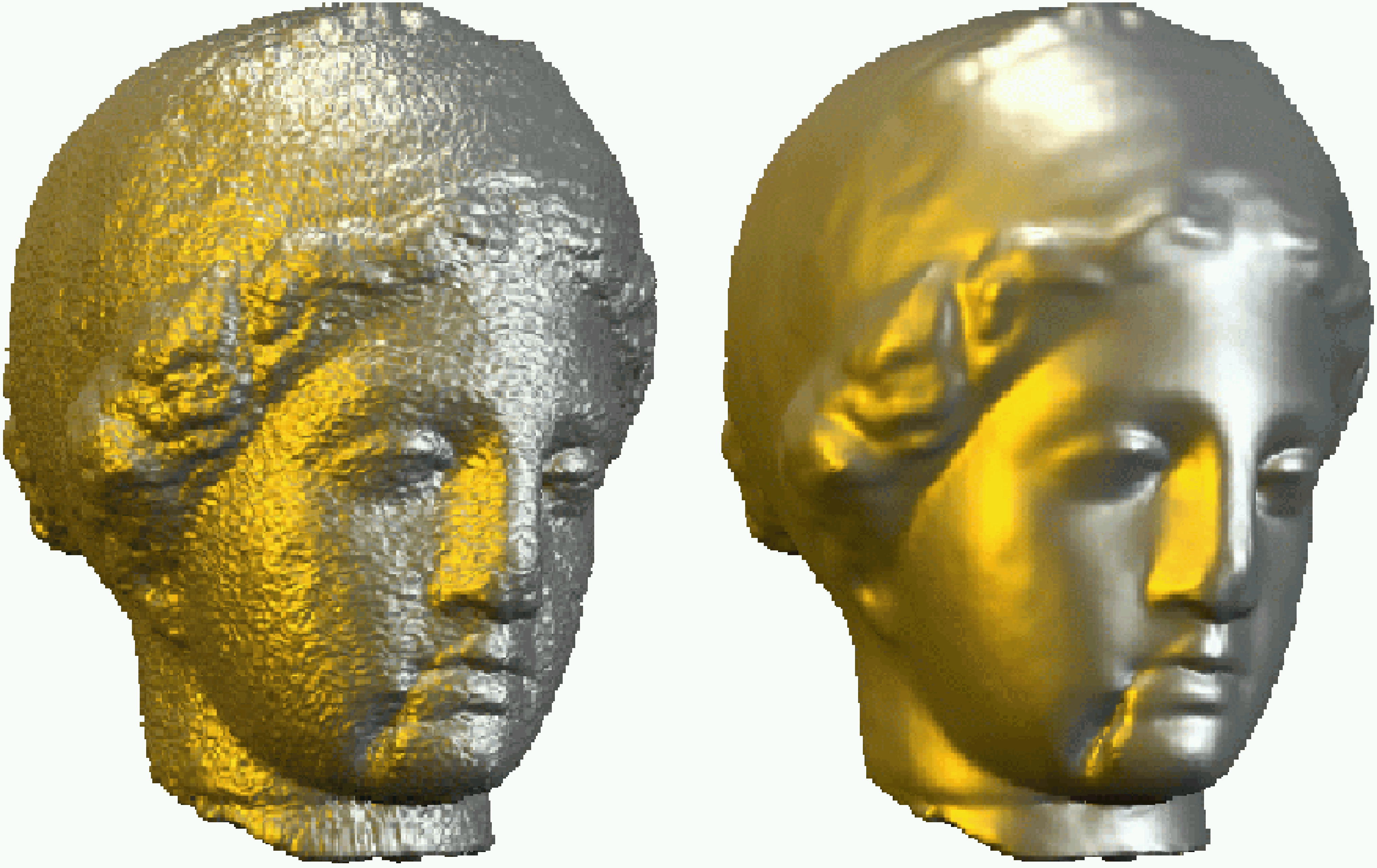


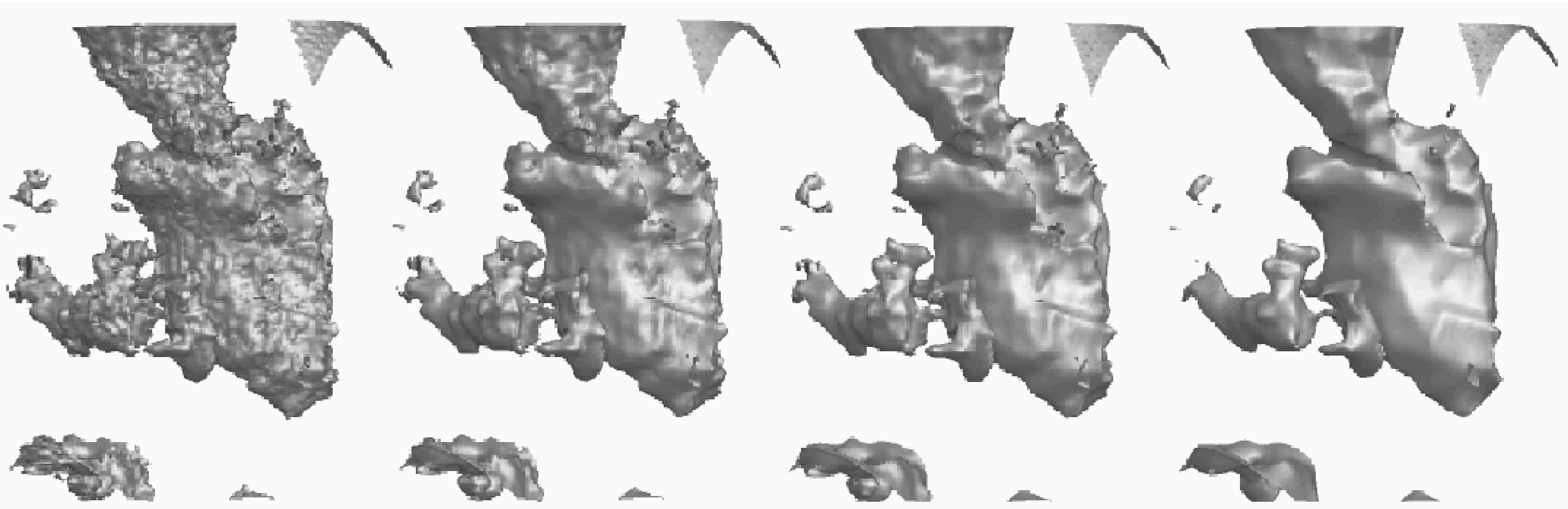


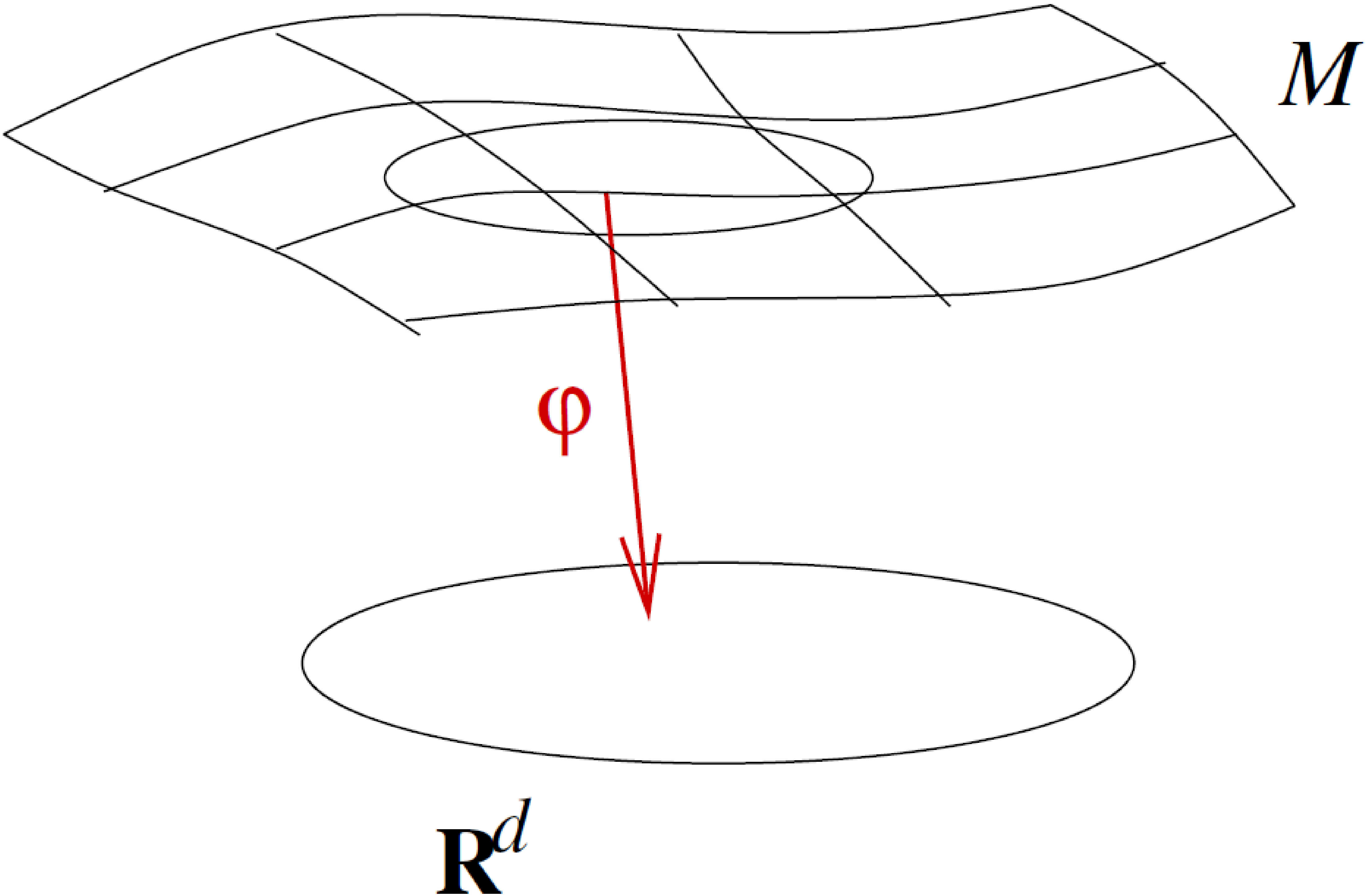




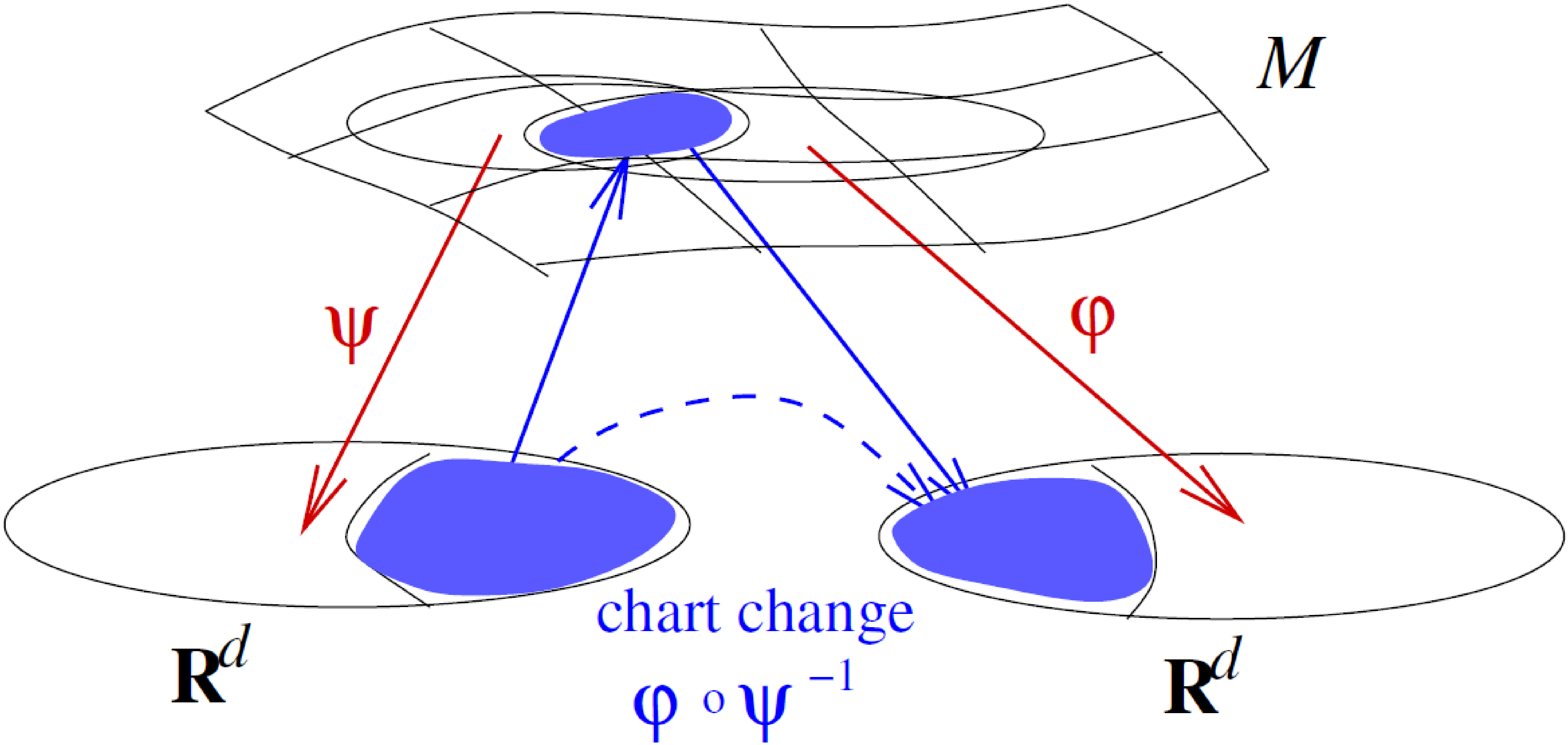








M
φ
$\mathbf{R}^d$


M
ψ
φ
chart change
$\varphi \circ \psi^{-1}$
$\mathbf{R}^d$
$\mathbf{R}^d$