Theoretical Foundations for Discrete Forward-and-Backward Diffusion Filtering

Martin Welk[1], Guy Gilboa[2], and Joachim Weickert[1]

[1] Mathematical Image Analysis Group
Faculty of Mathematics and Computer Science, Campus E1.1
Saarland University, 66041 Saarbrücken, Germany
{welk,weickert}@mia.uni-saarland.de
http://www.mia.uni-saarland.de

[2] 3DV Systems, 2nd Carmel St., Industrial Park Building 1
P. O. Box 249, Yokneam, 20692, Israel
gilboa@3dvsystems.com

Abstract. Forward-and-backward (FAB) diffusion is a method for sharpening blurry images (Gilboa et al. 2002). It combines forward diffusion with a positive diffusivity and backward diffusion where negative diffusivities are used. The well-posedness properties of FAB diffusion are unknown, and it has been observed that standard discretisations can violate a maximum-minimum principle. We show that for a novel nonstandard space discretisation which pays specific attention to image extrema, one can apply a modification of the space-discrete well-posedness and scale-space framework of Weickert (1998). This allows to establish well-posedness and a maximum-minimum principle for the resulting dynamical system. In the fully discrete 1-D case with an explicit time discretisation, a maximum-minimum principle and total variation reduction are proved in spite of the fact that negative diffusivities may appear. This provides a theoretical justification for applying FAB diffusion to digital images.

1 Introduction

In the last two decades, many partial differential equations (PDEs) and variational approaches have been proposed for enhancing digital images; see e.g. [1, 13] for an overview. The continuous framework behind these models offer advantages such as transparent and compact formulations where rotationally invariant approaches are easy to model.

However, some of the most interesting models are difficult to analyse in the continuous setting due to well-posedness problems. Often these filters work well in practice, but lack a sound continuous theory. This has triggered researchers to investigate well-posedness properties for space-discrete and fully discrete formulations. Let us mention a few examples.

For the Perona–Malik filter, Weickert [13] has proposed a space-discrete and fully discrete theory for smooth nonnegative diffusivities. Moreover, in [14] it is proven that the corresponding explicit scheme preserves monotonicity in the 1-D case. This explains that staircasing is the worst phenomenon that can happen. Pollak et al. [12] have extended this analysis to singular nonnegative diffusivities by showing well-posedness for dynamical systems with a discontinuous right hand sides that result from a space-discrete Perona-Malik model.

For the stabilised inverse linear diffusion process introduced by Osher and Rudin, it was not possible to establish a continuous well-posedness theory, but a stable minmod discretisation proved to work well in practice [9]. Later on, Breuß and Welk [2] showed that staircasing cannot be avoided by suitable space discretisations.

Shock filtering [5, 10] constitutes another example of a PDE that is difficult to analyse in the continuous setting, while for a 1-D space discretisation, Welk et al. [15] have shown that this process is well-posed and satisfies a maximum–minimum principle. It was even possible to find an analytic solution of the corresponding dynamical system.

On the variational side, Nikolova has published a number of impressive papers that provide deep insights in the behaviour of minimisers of space-discrete energies, even if they are highly nonconvex or nondifferentiable; see e.g. [7, 8]. It would have been extremely difficult if not impossible to obtain similar results in the continuous setting.

One PDE that has been proposed for sharpening images and for which no well-posed results are known so far, is the so-called *forward-and-backward (FAB) diffusion model* of Gilboa et al. [3]. Essentially this is a filter of Perona-Malik type, but its diffusivities are positive in certain areas and negative in others. Since pure inverse diffusion with a negative diffusivity is a prototype of an ill-posed problem, it is not surprising that no well-posedness results exist in the continuous setting. Experimentally it has been observed that straightforward explicit discretisations can violate a maximum–minimum principle.

The goal of our paper is to address this problem. We show that space-discrete FAB diffusion is well-posed and satisfies a maiximum–minimum principle if a specific non-standard discretisation is applied at extrema. This is achieved by modifying the space-discrete diffusion framework of Weickert [13]. Moreover, for the fully discrete 1-D case with an explicit time discretisation, a maximum-minimum principle and a total variation reduction property are established.

Our paper is organised as follows. In Section 2 we discuss the FAB diffusion model, while Section 3 reviews the space-discrete diffision framework from [13]. In the fourth section we present our nonstandard space discretisation for FAB diffusion, and we modify the space-discrete diffusion framework such that it becomes applicable to this model. The fully discrete 1-D case is discussed in detail in Section 5. Our paper is concluded with a summary in Section 6.

2 Forward-and-Backward Diffusion Filtering

Forward-and-backward (FAB) diffusion filtering has been introduced by Gilboa, Sochen and Zeevi in 2002 [3]. Let $\Omega \in \mathbb{R}^2$ be a rectangular image domain and consider a greyscale image $f : \Omega \to \mathbb{R}$ that is to be sharpened. Then FAB diffusion filtering creates filtered versions $u(\boldsymbol{x}, t)$ of $f(\boldsymbol{x})$ by solving a Perona-Malik type [11] equation

$$\partial_t u \ = \ \operatorname{div}\left(g(|\boldsymbol{\nabla} u|^2)\, \nabla u\right) \tag{1}$$

with f as initial condition,

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}), \tag{2}$$

and homogeneous Neumann boundary conditions,

$$\partial_{\boldsymbol{n}} u = 0, \tag{3}$$

where $\boldsymbol{n}$ denotes a normal vector to the image boundary $\partial\Omega$. Here $\boldsymbol{x} := (x, y)^\top$, subscripts denote partial derivatives, $\boldsymbol{\nabla} := (\partial_x, \partial_y)^\top$ is the spatial gradient, and div its corresponding divergence operator.

The diffusivity g may have different formulations, for example [4]:

$$g(s^2) = \frac{1}{\sqrt{1 + (s/k_f)^2}} - \frac{\alpha}{1 + (s/k_b)^2}, \tag{4}$$

where k_f and k_b control the gradient magnitudes for forward and backward diffusion, respectively, and α is the weight between these terms. Note that for small image gradients, this diffusivity is positive, while it becomes negative for larger ones, and finally becomes positive again. Our theory relies on the essential assumption $g(0) > 0$, which ensures that extrema undergo forward diffusion.

FAB diffusion has also been interpreted as an energy minimisation process of a nonmonotone potential in the shape of a triple-well [4]. In the variational formulation of [4] two additional terms have been introduced: a fidelity term to the input image and a fourth order term (hyper-diffusion) which increases the regularisation, strongly suppressing highly oscillating regions. Here we keep the notion of a sharpening flow without these terms. Connections between FAB diffusion and wavelet methods for image enhancement have been described in [6].

Apart from these results not many theoretical properties of the FAB process have been proven. In particular, existence, uniqueness and stability results are not available. Moreover, it was conjectured that such a process violates a maximum–minimum principle, as it may have a negative diffusivity [3]. This was shown to happen in numerical experiments, using standard numerical methods. In this paper we will prove that using a more sophisticated space discretisation, the process admits the maximum–minimum principle and useful theoretical results can be established.

3 A Space-Discrete Diffusion Framework

Let us now review the space-discrete diffusion framework of Weickert [13], since parts of it can be extended to the FAB setting. A standard discretisation of a Perona-Malik type diffusion equation

$$\partial_t u \ = \ \partial_x \left(g(|\boldsymbol{\nabla} u|^2)\, \partial_x u\right) \ + \ \partial_y \left(g(|\boldsymbol{\nabla} u|^2)\, \partial_y u\right) \tag{5}$$

in some inner pixel (i,j) yields the ordinary differential equation

$$\frac{du_{i,j}}{dt} = \frac{1}{h_1}\left(\frac{g_{i+1,j}+g_{i,j}}{2}\,\frac{u_{i+1,j}-u_{i,j}}{h_1} - \frac{g_{i,j}+g_{i-1,j}}{2}\,\frac{u_{i,j}-u_{i-1,j}}{h_1}\right) + \frac{1}{h_2}\left(\frac{g_{i,j+1}+g_{i,j}}{2}\,\frac{u_{i,j+1}-u_{i,j}}{h_2} - \frac{g_{i,j}+g_{i,j-1}}{2}\,\frac{u_{i,j}-u_{i,j-1}}{h_2}\right). \quad (6)$$

Here $u_{i,j}$ denotes an approximation to u in pixel (i,j). It is centred in the location $((i-\frac{1}{2})h_1, (j-\frac{1}{2})h_2)$, where h_1 and h_2 denote the grid size (pixel width) in x- resp. y-direction. This formula even holds for boundary pixels, provided that the homogeneous Neumann boundary conditions (3) are implemented by mirroring boundary pixels into dummy pixels. A suitable discretisation for the diffusivity g will be discussed later.

In a more compact notation, one can represent a pixel (i,j) by a single index $k(i,j)$. This leads to

$$\frac{du_k}{dt} = \sum_{n=1}^{2}\sum_{l\in\mathcal{N}_n(k)} \frac{g_l+g_k}{2h_n^2}\,(u_l-u_k), \quad (7)$$

where $\mathcal{N}_n(k)$ are the neighbours of pixel k in n-direction (boundary pixels may have less neighbours). This can be written as a system of ordinary differential equations (ODEs):

$$\frac{d\boldsymbol{u}}{dt} = A(\boldsymbol{u})\,\boldsymbol{u}, \quad (8)$$

where $\boldsymbol{u} = (u_1,...,u_N)^\top$, and the $N\times N$ matrix $A(\boldsymbol{u}) = (a_{k,l}(\boldsymbol{u}))$ satisfies

$$a_{k,l} := \begin{cases} \frac{g_k+g_l}{2h_n^2} & (l\in\mathcal{N}_n(k)), \\ -\sum_{n=1}^{2}\sum_{l\in\mathcal{N}_n(k)} \frac{g_k+g_l}{2h_n^2} & (l=k), \\ 0 & (\text{else}). \end{cases} \quad (9)$$

Denoting the index set $\{1,...,N\}$ by J, a space-discrete problem class (P_s) is defined in the following way.

$$\left.\begin{array}{l} \text{Let } \boldsymbol{f}\in\mathbb{R}^N. \text{ Find a function } \boldsymbol{u}\in \mathrm{C}^1([0,\infty),\mathbb{R}^N) \text{ that satisfies the} \\ \text{initial value problem} \\ \qquad\qquad \dfrac{d\boldsymbol{u}}{dt} = A(\boldsymbol{u})\,\boldsymbol{u}, \\ \qquad\qquad \boldsymbol{u}(0) = \boldsymbol{f}, \\ \text{where } A=(a_{ij}) \text{ has the following properties:} \\ \text{(S1)} \quad \text{Lipschitz-continuity of } A\in \mathrm{C}(\mathbb{R}^N,\mathbb{R}^{N\times N}) \text{ for every bounded} \\ \qquad\quad \text{subset of } \mathbb{R}^N, \\ \text{(S2)} \quad \text{symmetry:} \quad a_{ij}(\boldsymbol{u}) = a_{ji}(\boldsymbol{u}) \quad \forall\, i,j\in J,\ \forall\,\boldsymbol{u}\in\mathbb{R}^N, \\ \text{(S3)} \quad \text{vanishing row sums:} \quad \sum_{j\in J} a_{ij}(\boldsymbol{u}) = 0 \quad \forall\, i\in J,\ \forall\,\boldsymbol{u}\in\mathbb{R}^N, \\ \text{(S4)} \quad \text{nonnegative off-diagonals:} \quad a_{ij}(\boldsymbol{u}) \ge 0 \quad \forall\, i\ne j,\ \forall\,\boldsymbol{u}\in\mathbb{R}^N, \\ \text{(S5)} \quad \text{irreducibility for all } \boldsymbol{u}\in\mathbb{R}^N. \end{array}\right\} \ (P_s)$$

One should remember that a matrix $A \in \mathbb{R}^{N \times N}$ is called irreducible if for any $i, j \in J$ there exist k_0,...,$k_r \in J$ with $k_0 = i$ and $k_r = j$ such that $a_{k_p k_{p+1}} \neq 0$ for $p = 0$,...,$r-1$. In other words: There is a way from pixel i to pixel j along which the diffusivities do not vanish.

Under these requirements the subsequent theorem is proven in [13]:

Theorem 1. (Properties of Space-Discrete Diffusion Filtering)
For the space-discrete filter class (P_s) the following statements are valid:
(a) (Well-Posedness)
For every $T > 0$ the problem (P_s) has a unique solution $\boldsymbol{u}(t) \in \mathrm{C}^1([0,T], \mathbb{R}^N)$. This solution depends continuously on the initial value and the right-hand side of the ODE system.
(b) (Maximum-Minimum Principle)
Let $a := \min_{j \in J} f_j$ and $b := \max_{j \in J} f_j$. Then, $a \leq u_i(t) \leq b$ for all $i \in J$ and $t \in [0, T]$.
(c) (Average Grey Level Invariance)
The average grey level $\mu := \frac{1}{N} \sum_{j \in J} f_j$ is not affected by the space-discrete diffusion filter: $\frac{1}{N} \sum_{j \in J} u_j(t) = \mu$ for all $t > 0$.
(d) (Lyapunov Functionals)
$V(t) := \Phi(\boldsymbol{u}(t)) := \sum_{i \in J} r(u_i(t))$ is a Lyapunov function for all $r \in \mathrm{C}^1[a,b]$ with increasing r' on $[a,b]$: $V(t)$ is decreasing and bounded from below by $\Phi(\boldsymbol{c})$, where $\boldsymbol{c} := (\mu, ..., \mu)^\top \in \mathbb{R}^N$.
(e) (Convergence to a Constant Steady State)
$\lim\limits_{t \to \infty} \boldsymbol{u}(t) = \boldsymbol{c}$.

The proof shows that not all of the requirements (S1)–(S5) are necessary for each of the theoretical results above: Requirement (S1) is needed for local well-posedness, while proving a maximum–minimum principle requires (S3) and (S4). Local well-posedness together with the maximum–minimum principle implies global well-posedness. The average grey value invariance is based on (S2) and (S3). The existence of Lyapunov functionals can be established by means of (S2)–(S4), and convergence to a constant steady state requires (S5) in addition to (S2)–(S4).

4 Application to Space-Discrete FAB Diffusion

It is straightforward to verify the prerequisites (S1)–(S5) for the popular *positive* diffusivity functions, such that Theorem 1 is applicable. However, for FAB diffusion negative diffusivities are possible and the situation becomes much more complicated. One immediatly sees that space-discrete FAB diffusion satisfies (S1: smoothness), (S2: symmetry), and (S3: vanishing row sums). However, this just implies local well-posedness and average grey level invariance.

By inspecting (9) it becomes clear that (S4: nonnegative off-diagonals) and (S5: irreducibility) cannot be satisfied for typical FAB diffusivities: These diffusivities may vanish (which violates (S5)) and they may even become negative (violating (S4)). As a consequence, global well-posedness, a maximum–minimum principle, Lyapunov functions and convergence to a constant steady state cannot be proven in this way.

For the practical applicability of FAB diffusion it would be highly desirable to have at least global well-posedness and a maximum–minimum principle. Is there a remedy for these properties? Fortunately the answer is affirmative, since (S4: nonnegative off-diagonals) can be replaced by a less restrictive condition that only holds at extrema:

Theorem 2. (Space-Discrete Diffusion Filtering under Weaker Conditions) *Assume that a space-discrete filter satisfies only the properties (S1)–(S3) of the framework (P_s), and*

(S4a) nonnegative off-diagonals at extrema:

$a_{i,j}(\boldsymbol{u}) \geq 0$ *for all* $j \in J$ *with* $j \neq i$ *if* $\boldsymbol{u}$ *has an extremum in* i.

Then the well-posedness result (a), the maximum–minimum principle (b), and the average grey level invariance (c) of Theorem 1 are still satisfied.

Proof. Following [13], one observes that in some pixel k that is a discrete global maximum (i.e. $u_k \geq u_j$ for all $j \in J$), condition (S4a) implies that

$$\begin{aligned} \frac{du_k}{dt} &= \sum_{j \in J} a_{kj}(\boldsymbol{u})\, u_j \\ &= a_{kk}(\boldsymbol{u})\, u_k + \sum_{j \in J \setminus \{k\}} \underbrace{a_{kj}(\boldsymbol{u})}_{\geq 0} \underbrace{u_j}_{\leq u_k} \\ &\leq u_k \cdot \sum_{j \in J} a_{kj}(\boldsymbol{u}) \\ &\overset{\text{(S3)}}{=} 0. \end{aligned} \tag{10}$$

In the same way one can prove that if k is a minimum, one has $\frac{du_k}{dt} \geq 0$.

This nonenhancement behaviour in extrema is the only place where nonnegativity is required in the entire proof of the maximum–minimum principle in [13]. As a consequence, the maximum–minimum principle still holds if (S4) is replaced by the weaker condition (S4a). Moreover, together with local well-posedness, global well-posedness is obtained. This completes the proof.

While the preceding results are encouraging, we have not yet shown that a suitable space-discretisation satisfies the nonnegativity requirement (S4a) at extrema. Unfortunately, this issue is a bit more delicate than one might assume: A standard discretisation of the diffusivity $g(|\nabla u|^2)$ in some pixel (i,j) is given by the central difference approximation

$$g_{i,j} := g\left(\left(\frac{u_{i+1,j} - u_{i-1,j}}{2h_1}\right)^2 + \left(\frac{u_{i,j+1} - u_{i,j-1}}{2h_2}\right)^2\right) \tag{11}$$

Note that even if $\boldsymbol{u}$ has an extremum in (i,j), the preceding central difference approximation of $|\nabla u|^2$ may become positive – and not 0 as one would expect from the continuous theory. Since the FAB diffusivities only guarantee that $g(0) > 0$, it can happen that this finite difference approximation creates negative diffusivities in extrema and (S4a) is violated. Fortunately there is an interesting alternative to the standard discretisation of the diffusivity that solves these problems immediately:

Theorem 3. (Properties of Space-Discrete FAB Diffusion)
The space discretisation (6) of FAB diffusion is well-posed, satisfies a maximum–minimum principle and average grey level invariance, if the diffusivity is evaluated by the nonstandard finite difference approximation

$$\begin{aligned} g_{i,j} \;:=\; g\, & \left(\max\left(\frac{u_{i+1,j}-u_{i,j}}{h_1} \cdot \frac{u_{i,j}-u_{i-1,j}}{h_1},\, 0 \right) \right. \\ & \left. + \max\left(\frac{u_{i,j+1}-u_{i,j}}{h_2} \cdot \frac{u_{i,j}-u_{i,j-1}}{h_2},\, 0 \right) \right). \end{aligned} \tag{12}$$

It should be noted that this approximation has the same quadratic order of consistency as the previous one. However, it guarantees a vanishing discrete gradient approximation in extrema. As a consequence, (S4a) is guaranteed, since FAB diffusities satisfy $g(0) > 0$. Interestingly, the property $g(0) > 0$ is the only requirement that is necessary in order to establish well-posedness and a maximum–minimum principle for space-discrete FAB diffusion.

Last but not least, these results are not restricted to the two-dimensional case: With a similar nonstandard approximation, it is straightforward to verify that space-discrete FAB diffusion is well-posed and satisfies an extremum principle in any dimension.

5 Fully Discrete FAB Diffusion

In order to establish useful properties for FAB diffusion in the fully discrete case, we restrict ourselves to the 1-D setting and use a simple explicit time discretisation with step size τ. Then the corresponding scheme to $\partial_t u = \partial_x (g((\partial_x u)^2)\, \partial_x u)$ is given by

$$\frac{u_i^{k+1}-u_i^k}{\tau} \;=\; \frac{g_{i-1}^k + g_i^k}{2} \cdot \frac{u_{i-1}^k - u_i^k}{h^2} + \frac{g_{i+1}^k + g_i^k}{2} \cdot \frac{u_{i+1}^k - u_i^k}{h^2} \tag{13}$$

with the nonstandard approximation $g_i^k = g\left(\max\left(\frac{u_i^k - u_{i-1}^k}{h} \cdot \frac{u_{i+1}^k - u_i^k}{h}, 0\right)\right)$. The upper index denotes the time level, i.e. u_i^k approximates u at location $(i-\frac{1}{2})h$ and time $k\tau$. This approximation also holds at the boundary pixels u_1 and u_N when one uses the before mentioned dummy pixels.

For our analysis, two additional assumptions are essential. While the first one refers to the range of grey values, the second one requires a diffusivity g that still takes sufficiently large positive values for small positive arguments. We get the following result.

Theorem 4. (Properties of Fully Discrete FAB Diffusion)
Let an initial 1-D image $\boldsymbol{f} = (f_i)$ be given and let the sequence of images $\boldsymbol{u}^k = (u_i^k)$ evolve according to (13) with the initial condition $\boldsymbol{u}^0 = \boldsymbol{f}$. Let the grey-values f_i be restricted to a finite interval of length R. Assume further that two constants $c_1 > c_2 > 0$ exist such that the diffusivity g fulfils $g(0) = c_1$, and $g(z) > -c_2$ for all $z > 0$. Moreover, assume that a positive ω exists such that $g(s^2) > c_2$ holds for all s with $0 < s < \omega R$.

If the time step satisfies

$$\tau < \frac{\omega^2 h^4}{c_1 + c_2 + 2c_1\omega^2 h^2} \ , \tag{14}$$

the following results are true for the evolution of $(\boldsymbol{u}^k)$.

(a) (Maximum–Minimum Principle)
If the initial signal is bounded by $a \le f_i \le b$ for all i, then $a \le u_i^k \le b$ holds for all i and all $k \ge 0$.

(b) (Total Variation Reduction)
For each time step $k \ge 0$, the total variation of the image $\boldsymbol{u}^{k+1}$ is less or equal to the total variation of $\boldsymbol{u}^k$:

$$\sum_{i=1}^{N-1} \left|u_{i+1}^{k+1} - u_i^{k+1}\right| \ \le \ \sum_{i=1}^{N-1} \left|u_{i+1}^{k} - u_i^{k}\right| \ . \tag{15}$$

Proof. The global statements of the theorem follow from local properties which will be proved in four steps.

Step 1: A local maximal pixel does not increase.
Assume that u_i^k is a local maximum of the 1-D image in time step k, i.e. we have $u_i^k \ge u_{i+1}^k$ and $u_i^k \ge u_{i-1}^k$. Since in this case $g_{i-1}^k + g_i^k$ and $g_i^k + g_{i+1}^k$ are certainly nonnegative, u_i^{k+1} is a convex combination of u_{i-1}^k, u_i^k and u_{i+1}^k if only

$$1 - \frac{\tau}{2h^2}(g_{i-1}^k + 2g_i^k + g_{i+1}^k) \ge 0 \tag{16}$$

holds. Because of $g_{i-1}^k + 2g_i^k + g_{i+1}^k \le 4c_1$ this is certainly the case if

$$\tau \le \frac{h^2}{2c_1} \ . \tag{17}$$

Step 2: A neighbour pixel of a local maximum remains below this maximum.
Assume that u_i^k is a maximum and u_{i+1}^k is not a local minimum. Then the inequality $u_{i+1}^{k+1} \le u_i^k$ holds if

$$\tau \le \frac{\omega^2 h^4}{2c_1\omega^2 h^2 + c_2} \ . \tag{18}$$

To see this, we use the equation

$$u_{i+1}^{k+1} = u_{i+1}^k + \tau \cdot \left(\frac{g_i^k + g_{i+1}^k}{2} \cdot \frac{u_i^k - u_{i+1}^k}{h^2} + \frac{g_{i+1}^k + g_{i+2}^k}{2} \cdot \frac{u_{i+2}^k - u_{i+1}^k}{h^2} \right) \quad (19)$$

and distinguish two cases.

Case 1: $(u_{i+1}^k - u_i^k)(u_{i+2}^k - u_{i+1}^k) \le \omega^2 h^2 R^2$.

Then $g_{i+1}^k + g_{i+2}^k$ is certainly nonnegative. The right-hand side of (19) is therefore a convex combination of u_i^k, u_{i+1}^k and u_{i+2}^k if (16) holds. Analogous to our above reasoning, this is true if (17) is satisfied.

Case 2: $(u_{i+1}^k - u_i^k)(u_{i+2}^k - u_{i+1}^k) > \omega^2 h^2 R^2$.

Here we conclude from $u_{i+1}^k - u_{i+2}^k \le R$ that

$$u_i^k - u_{i+1}^k > \omega^2 h^2 R \; . \quad (20)$$

Using $\frac{1}{2}(g_i^k + g_{i+1}^k) < c_1$ and $\frac{1}{2}(g_{i+1}^k + g_{i+2}^k) > -c_2$ we obtain from (19) the estimate

$$u_{i+1}^{k+1} \le u_{i+1}^k + \frac{\tau}{h^2} c_1 (u_i^k - u_{i+1}^k) + \frac{\tau}{h^2}\, c_2\, R \quad (21)$$

which ensures $u_{i+1}^{k+1} \le u_i^k$, provided that

$$\tau \le \frac{\omega^2 h^4}{c_1 \omega^2 h^2 + c_2} \quad (22)$$

holds.

Condition (18) ensures the bounds of both cases, i.e. (17) and (22).

Step 3: No new extrema are generated around existing extrema.
Assume that u_i^k is a local maximum, and none of its neighbours is a local minimum. Assume first that

$$(u_{i+1}^k - u_i^k)(u_{i+2}^k - u_{i+1}^k) > \omega^2 R^2 \quad (23)$$

and thus again (20) and (21) hold.

Similar considerations for u_i^{k+1} yield

$$u_i^{k+1} \ge u_i^k + \frac{\tau}{h^2} c_1 (u_{i+1}^k - u_i^k) - \frac{\tau}{h^2} c_1 R \quad (24)$$

which together with (21) implies

$$u_i^{k+1} - u_{i+1}^{k+1} \ge \left(1 - 2\frac{\tau}{h^2} c_1\right)(u_i^k - u_{i+1}^k) - \frac{\tau}{h^2}(c_1 + c_2) R \; . \quad (25)$$

By the hypothesis of the theorem, (14), and (20) we have that

$$\tau < \frac{h^2}{(c_1 + c_2) R / (u_i^k - u_{i+1}^k) + 2 c_1} \; , \quad (26)$$

such that the expression on the right-hand side of (25) is nonnegative.

Therefore u_{i+1}^{k+1} can become a maximum in $(\boldsymbol{u}^{k+1})$ only if

$$(u_{i+1}^k - u_i^k)(u_{i+2}^k - u_{i+1}^k) \le \omega^2 h^2 R^2 \ . \tag{27}$$

Analogous reasoning applies to the left neighbour u_{i-1}^{k+1}. This means that the maximum property of pixel i can be shifted to one of its neighbours. Our assertion that no new extrema are generated remains true except if both neighbours u_{i-1}^{k+1} and u_{i+1}^{k+1} would simultaneously turn into maxima.

Let us therefore discuss this case. This would require the two inequalities

$$\begin{aligned} & (u_{i+1}^k - u_i^k)(u_{i+2}^k - u_{i+1}^k) \le \omega^2 h^2 R^2 \\ \text{and} \quad & (u_{i-1}^k - u_{i-2}^k)(u_i^k - u_{i-1}^k) \le \omega^2 h^2 R^2 \end{aligned} \tag{28}$$

to hold at the same time. In this situation, however, $g_{i+1}^k + g_{i+2}^k$ and $g_{i-1}^k + g_{i-2}^k$ are nonnegative, implying

$$\begin{aligned} & u_{i+1}^{k+1} \le u_{i+1}^k + \tau c_1 \frac{u_i^k - u_{i+1}^k}{h^2} \\ \text{and} \quad & u_{i-1}^{k+1} \le u_{i-1}^k + \tau c_1 \frac{u_i^k - u_{i-1}^k}{h^2} \ , \end{aligned} \tag{29}$$

while for the central pixel

$$u_i^{k+1} \ge u_i^k + \tau c_1 \frac{u_{i-1}^k - 2u_i^k + u_{i+1}^k}{h^2} \tag{30}$$

holds. Hence,

$$-u_{i-1}^{k+1} + 2u_i^{k+1} - u_{i+1}^{k+1} \ge \left(1 - 2\frac{\tau}{h^2} c_1\right)(-u_{i-1}^k + 2u_i^k - u_{i+1}^k) \ . \tag{31}$$

For $\tau \le \frac{h^2}{2c_1}$, the right-hand side is clearly nonnegative which ensures that u_{i-1}^{k+1} and u_{i+1}^{k+1} cannot both become maxima.

Step 4: Monotonicity is preserved in image segments without extrema.
Assume that $u_i^k > u_{i+1}^k > u_{i+2}^k > u_{i+3}^k$. We show that then also $u_{i+1}^{k+1} \ge u_{i+2}^{k+1}$ holds. In the proof we distinguish three cases.

Case 1: $g_i^k + g_{i+1}^k \ge 0$ *and* $g_{i+2}^k + g_{i+3}^k \ge 0$.

Then

$$u_{i+1}^{k+1} - u_{i+2}^{k+1} \ge \left(1 - 2\frac{\tau}{h^2} c_1\right)(u_{i+1}^k - u_{i+2}^k) \tag{32}$$

such that the right-hand side is again nonnegative if (17) holds.

Case 2: $g_i^k + g_{i+1}^k \geq 0$ *and* $g_{i+2}^k + g_{i+3}^k < 0$.

(The case $g_i^k + g_{i+1}^k < 0$ and $g_{i+2}^k + g_{i+3}^k \geq 0$ is treated in a symmetric way.)

From $u_{i+2}^k - u_{i+3}^k \leq R$ and $(u_{i+1}^k - u_{i+2}^k)(u_{i+2}^k - u_{i+3}^k) > \omega^2 h^2 R^2$ we obtain

$$u_{i+1}^k - u_{i+2}^k > \omega^2 h^2 R \; . \tag{33}$$

Consequently,

$$\begin{aligned} u_{i+1}^{k+1} - u_{i+2}^{k+1} &\geq u_{i+1}^k - u_{i+2}^k - 2\frac{\tau}{h^2} c_1 (u_{i+1}^k - u_{i+2}^k) - \frac{\tau}{h^2} c_2 (u_{i+2}^k - u_{i+3}^k) \\ &> u_{i+1}^k - u_{i+2}^k - 2\frac{\tau}{h^2} c_1 (u_{i+1}^k - u_{i+2}^k) - \frac{\tau}{h^2} c_2 R \; . \end{aligned} \tag{34}$$

Due to (33) the right-hand side is certainly nonnegative if

$$\tau \leq \frac{\omega^2 h^4}{2 c_1 \omega^2 h^2 + c_2} \; . \tag{35}$$

Case 3: $g_i^k + g_{i+1}^k < 0$ *and* $g_{i+2}^k + g_{i+3}^k < 0$.

Since in this case we have

$$(u_i^k - u_{i+1}^k) + (u_{i+2}^k - u_{i+3}^k) \leq R \; , \tag{36}$$

it follows that

$$(u_{i+1}^k - u_{i+2}^k) \min(u_i^k - u_{i+1}^k, u_{i+2}^k - u_{i+3}^k) > \omega^2 h^2 R^2 \tag{37}$$

and thus

$$u_{i+1}^k - u_{i+2}^k > 2\omega^2 h^2 R \; . \tag{38}$$

A similar reasoning as in Case 2 gives that $u_{i+1}^{k+1} - u_{i+2}^{k+1}$ is ensured if

$$\tau \leq \frac{\omega^2 h^4}{2 c_1 \omega^2 h^2 + c_2/2} \; . \tag{39}$$

Comparing the bounds derived for the different statements yields (14) as the most restrictive one. If this condition is imposed, extrema cannot be created but only shifted to neighbouring pixels, and monotone segments preserve their monotonicity. Both the maximum–minimum principle and the reduction of total variation follow immediately. This completes the proof.

We are convinced that Theorem 4 also possesses a 2-D analogue. The preceding proof, however, does not transfer in a straightforward way to this case: The dependency of g on nonstandard discretisations of u_x and u_y (cf. (12)) makes it highly cumbersome to control the sign of g.

6 Summary and Conclusions

In spite of its negative diffusivity, FAB diffusion becomes well-posed if a nonstandard space discretisation is used. It guarantees a positive diffusivity in discrete extrema. This result is fundamental for justifying FAB diffusion in a practical setting with digital images. Our ongoing work includes research on the multidimensional fully discrete case as well as extensions of our results to (semi-)implicit time discretisations.

Acknowledgement. This work has been initiated during a visit of Guy Gilboa to Saarland University. His visit has been financially supported by the Minerva Foundation.

References

1. Aubert, G., Kornprobst, P.: Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations. Second edn. Volume 147 of Applied Mathematical Sciences. Springer, New York (2006)
2. Breuß, M., Welk, M.: Staircasing in semidiscrete stabilised inverse diffusion algorithms. Journal of Computational and Applied Mathematics **206** (2007) 520–533
3. Gilboa, G., Sochen, N.A., Zeevi, Y.Y.: Forward-and-backward diffusion processes for adaptive image enhancement and denoising. IEEE Transactions on Image Processing **11**(7) (2002) 689–703
4. Gilboa, G., Sochen, N., Zeevi, Y.Y.: Image sharpening by flows based on triple well potentials. Journal of Mathematical Imaging and Vision **20** (2004) 121–131
5. Kramer, H.P., Bruckner, J.B.: Iterations of a non-linear transformation for enhancement of digital images. Pattern Recognition **7** (1975) 53–58
6. Mrázek, P., Weickert, J., Steidl, G.: Diffusion-inspired shrinkage functions and stability results for wavelet denoising. International Journal of Computer Vision **64**(2/3) (September 2005) 171–186
7. Nikolova, M.: Local strong homogeneity of a regularized estimator. SIAM Journal on Applied Mathematics **61**(2) (2000) 633–658
8. Nikolova, M.: Minimizers of cost-functions involving nonsmooth data fidelity terms. application to the processing of outliers. SIAM Journal on Numerical Analysis **40**(3) (2002) 965–994
9. Osher, S., Rudin, L.: Shocks and other nonlinear filtering applied to image processing. In Tescher, A.G., ed.: Applications of Digital Image Processing XIV. Volume 1567 of Proceedings of SPIE. SPIE Press, Bellingham (1991) 414–431
10. Osher, S., Rudin, L.I.: Feature-oriented image enhancement using shock filters. SIAM Journal on Numerical Analysis **27** (1990) 919–940
11. Perona, P., Malik, J.: Scale space and edge detection using anisotropic diffusion. IEEE Transactions on Pattern Analysis and Machine Intelligence **12** (1990) 629–639
12. Pollak, I., Willsky, A.S., Krim, H.: Image segmentation and edge enhancement with stabilized inverse diffusion equations. IEEE Transactions on Image Processing **9**(2) (February 2000) 256–266
13. Weickert, J.: Anisotropic Diffusion in Image Processing. Teubner, Stuttgart (1998)
14. Weickert, J., Benhamouda, B.: A semidiscrete nonlinear scale-space theory and its relation to the Perona–Malik paradox. In Solina, F., Kropatsch, W.G., Klette, R., Bajcsy, R., eds.: Advances in Computer Vision. Springer, Wien (1997) 1–10
15. Welk, M., Weickert, J., Galić, I.: Theoretical foundations for spatially discrete 1-D shock filtering. Image and Vision Computing **25**(4) (2007) 455–463