Families of Generalised Morphological Scale Spaces

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Abstract. Morphological and linear scale spaces are well-established instruments in image analysis. They display interesting analogies which make a deeper insight into their mutual relation desirable. A contribution to the understanding of this relation is presented here.
We embed morphological dilation and erosion scale spaces with paraboloid structure functions into families of scale spaces which are found to include linear Gaussian scale space as limit cases. The scale-space families are obtained by deforming the algebraic operations underlying the morphological scale spaces within a family of algebraic operations related to l^p norms and generalised means. Alternatively, the deformation of the morphological scale spaces can be described in terms of grey-scale isomorphisms.
We discuss aspects of the newly constructed scale space families such as continuity, invariance, and separability, and the limiting procedure leading to linear scale space. This limiting procedure requires a suitable renormalisation of the scaling parameter. In this sense, our approach turns out to be complementary to that proposed by L. Florack et al. in 1999 which comprises a continuous deformation of linear scale space including morphological scale spaces as limit cases provided an appropriate renormalisation.
Keywords: morphological scale space, linear scale space, dilation, erosion, deformation

1 Introduction

A scale space [10, 11, 1–3, 5, 6, 13, 18, 15] can be described as a family of filters which transform a given signal into a simplified signal. The family of operators is equipped with a linear ordering, from fine to coarse resolution, having the identity as minimal element, and is required to fulfil the causality condition, i.e. structural details of the signal such as extrema must not be enhanced under the filter action. For a scale space in strict sense it is also required that concatenation of filter operators is equivalent to one single filter operator of coarser resolution, such that the filter operators form a semi-group.

An outstanding example with multiple applications is the Gaussian linear scale space made up of convolutions of the original signal with Gaussians of increasing standard deviation [10, 11, 5]. Since a two-dimensional Gaussian can be written as the product of two one-dimensional Gaussians, this scale space is separable. Here, separability means that the filtering in two dimensions can be performed by filtering in x and y directions subsequently, and it is a highly desirable property particularly from the computational point of view. The semigroup property holds because the convolution of Gaussians is again a Gaussian with the sum of variances. Last but not least, Gaussian filtering is rotationally invariant which is an important requirement particularly in image processing applications.

Morphological dilation and erosion, with structure functions of increasing size in the scaling parameter, define another class of scale spaces [4, 16, 17, 12]. Provided that a rotation-symmetric quadratic structure function is used, one obtains again rotational invariance, separability and semigroup property.

Besides sharing many useful properties making them valuable for denoising and other image analysis applications, both before-mentioned classes of scale spaces display also similarities in their structure which have been noted e.g. in [16, 17, 8, 7]. The defining formulas of dilation and erosion can formally be obtained from that of convolution by replacing addition with maximum or minimum, and multiplication with addition. Now the real numbers equipped with maximum and addition form a semi-ring, the so-called max-plus algebra, cf. e.g. [14]. Being only a semi-ring, the max-plus algebra is a weaker algebraic structure than the usual, plus-product algebra, but still has many parallels to the latter. The most important difference is the lacking of a neutral and inverse for maximum. Quite alike, there is also a min-plus algebra. Essentially, dilation and erosion are in the max-plus and min-plus algebras what convolution is in the plus-product algebra. Furthermore, convolution in plus-product algebra is in close relation to the Fourier transform which carries convolution to multiplication and vice versa. In max-plus algebra, there is the slope transform which stands in mostly the same relation to dilation: it carries over dilation to addition and vice versa [4]. We shall not pursue the latter analogy but concentrate on the scale spaces instead.

Starting from the observation that all algebraic operations involved – addition, multiplication, minimum/maximum – fit into one single parametrised family of operations which stands in close relation to generalised means and l^p norms, we describe a variety of scale spaces which in some sense interpolate between morphological and linear scale spaces.

Parametrised families of scale spaces that allow a continuous transition, in some sense, between different fundamental scale spaces have already been proposed in the literature, see [8, 7, 5, 9]. While in [5] Poisson and Gaussian scale spaces are considered, the construction by Florack et al. from [8, 7] is of particular interest for us since it is also concerned with linking morphological and Gaussian scale spaces. The scaling procedure used in [9] also includes the grey-value transformations via power functions that can be used to describe subsets of the filter families discussed here, see section 2.4.

The paper is organised as follows: In paragraph 2.1 we introduce the algebraic operations to be used, and we collect some basic facts about them. In paragraph 2.2 we define the family of scale spaces that are treated in this paper. These objects are studied in more detail then. While paragraph 2.3 contains limit statements securing the continuity of the family of scale spaces as a whole, the properties of the individual scale spaces are investigated in paragraph 2.4. A comparison to the family of scale spaces proposed by Florack et al. is given in paragraph 2.5. In section 3, the interpolation property of our scale space family is illustrated with an example picture.

2 Generalised morphological scale spaces

2.1 A family of algebraic operations

First we introduce algebraic operations and integrals for later use. Throughout the paper, $\mathbb{R}_0^+$ and $\mathbb{R}^+$ denote non-negative and positive real numbers, resp.

Definition 1. *Let $\varphi : R \to R'$ be a continuous, monotonic, one-to-one function where each of R and R' may stand for $\mathbb{R}$, $\mathbb{R}_0^+$ or $\mathbb{R}^+$. Then we define*

$$a +_\varphi b := \varphi^{-1}(\varphi(a) + \varphi(b)) \tag{1}$$

and call it φ-deformed addition. Analogously, we define the φ-deformed integral of a function f over a domain $D \subset \mathbb{R}^n$ by

$$_\varphi\!\int_D f(x)\,\mathrm{d}x := \varphi^{-1}\left(\int_D \varphi(f(x))\,\mathrm{d}x\right). \tag{2}$$

Given a second continuous, monotonic, one-to-one function $\psi : R \to R''$, $R'' \in \{\mathbb{R}, \mathbb{R}_0^+, \mathbb{R}^+\}$, we call

$$(f *_{\varphi,\psi} g)\,(x) := {}_\varphi\!\int_{\mathbb{R}} f(x-y) +_\psi g(y)\,\mathrm{d}x \tag{3}$$

(φ, ψ)-deformed convolution.

Keeping this in mind, we turn to have a – rather grazing – look at generalised means.

Definition 2. *Assume $p \in \mathbb{R} \setminus \{0\}$. For $a, b \in \mathbb{R}^+$, or even $a, b \in \mathbb{R}_0^+$ if $p > 0$, let*

$$M_p(a,b) := \left(\frac{a^p + b^p}{2}\right)^{1/p}. \tag{4}$$

Moreover, let for $a, b \in \mathbb{R}_0^+$

$$M_{-\infty}(a,b) := \min(a,b),\ M_0(a,b) := (ab)^{1/2},\ M_{+\infty}(a,b) := \max(a,b). \tag{5}$$

For $p \in \mathbb{R} \cup \{\pm\infty\}$, $M_p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is called p-th generalised mean.

It is well-known that for $a, b \in \mathbb{R}^+$, $M_p(a, b)$ as a function in $p \in \mathbb{R} \cup \{\pm\infty\}$ is continuous and monotonically increasing everywhere. If $a \neq b$, monotony is strict. In this way the geometric mean fits smoothly into the series of power means, as do maximum and minimum as limit cases. This fact motivates us to interpolate between the three algebraic operations multiplication, addition and maximum in the following way.

Definition 3. *Let $p \in \mathbb{R} \setminus \{0\}$. For $a, b \in \mathbb{R}^+$, or even $a, b \in \mathbb{R}_0^+$ provided that p is positive, define*

$$a +_p b := (a^p + b^p)^{1/p} . \tag{6}$$

Further let for $a, b \in \mathbb{R}_0^+$

$$a +_{-\infty} b := \min(a, b),\ \ a +_0 b := ab,\ \ a +_{+\infty} b := \max(a, b). \tag{7}$$

Note that $+_p$, $p \in \mathbb{R}$ is exactly the φ-deformed addition in the sense of definition 1 if $\varphi(x) = x^p$ for $p \neq 0$, $\varphi(x) = \ln x$ for $p = 0$. Besides this, $+_p$ for $p \in [1, +\infty)$ is just the l^p-norm of the finite sequence (a, b).

It is obvious that all $+_p$, $p \in \mathbb{R} \cup \{\pm\infty\}$, are commutative and associative. Distributivity between two of these operations, $(a +_q b) +_p c = (a +_p c) +_q (b +_p c)$ for all admissible a, b, c, holds if and only if $q = \pm\infty$ or $p = 0$.

For non-negative real p or $p = +\infty$ it makes sense to define a partially inverse operation for $+_p$ in the following way:

Definition 4. *For $p \in [0, +\infty]$, $a, b \in \mathbb{R}_0^+$, we define $a -_p b := \inf\{c \in \mathbb{R}_0^+ \mid c +_p b \geq a\}$.*

For $p = 0$, $-_p$ coincides with division for all $a \geq 0$, $b > 0$. If $p > 0$, $(a -_p b) +_p b = a$ holds only for $a \geq b$. It is clear that in this case one can calculate $a -_p b = (a^p - b^p)^{1/p}$ while $a -_p b = a$ for $p = +\infty$.

Using the same deformation functions φ as for $+_p$, we can also introduce modified integrals.

Definition 5. *For continuous functions $f : \mathbb{R}^n \to \mathbb{R}_0^+$ and domains $D \subset \mathbb{R}^n$ we define the p-integral by*

$$ {}_p\!\int_D f(x)\,\mathrm{d}x := \left(\int_D (f(x))^p \,\mathrm{d}x \right)^{1/p} \qquad \text{for } p \in \mathbb{R} \setminus \{0\}, \tag{8}$$

$$ {}_0\!\int_D f(x)\,\mathrm{d}x := \exp \int_D \ln f(x)\,\mathrm{d}x, \tag{9}$$

$$ {}_{+\infty}\!\int_D f(x)\,\mathrm{d}x := \sup_{x \in D} f(x), \qquad {}_{-\infty}\!\int_D f(x)\,\mathrm{d}x := \inf_{x \in D} f(x). \tag{10}$$

For $p \in [1, +\infty]$ these p-integrals coincide with the $L^p(D)$-norms of f.

As in definition 1, generalised addition and integral can be combined to form a (q, p)-convolution of two functions. We refrain from carrying this out in a formal expression at this point; we shall use the idea in a slightly modified manner when introducing (q, p)-dilation.

2.2 Definition of generalised dilation and erosion scale spaces

We write a scale space as a family $\{F_t \mid t \in \mathbb{R}_0^+\}$ of mappings of some function space $\mathcal{F}$ over $\mathbb{R}^n$ into itself, with F_0 being the identity. The causality condition states that for any given function $u_0(x) = f(x)$ from this function space and any $t > 0$, the function $u_t(x) = F_t f(x)$ contains no details which are not contained in $u_{t'} = F_{t'} f(x)$ for all $0 \le t' \le t$.

Throughout the following, it is understood that $\mathcal{F}$ consists of the continuous, bounded functions over $\mathbb{R}^n$ with compact support. Since $f \in \mathcal{F}$ is to represent a given image, we shall also assume that the range of f, representing grey values, is contained in $[0, 1]$.

Gaussian convolution linear scale space is given by

$$F_t f(x) = \int_{\mathbb{R}^n} f(x-y)\phi_{\sqrt{t}}(y)\,\mathrm{d}y, \quad \phi_\sigma(y) = \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(-\frac{\|y\|}{2\sigma^2}\right). \tag{11}$$

The morphological scale spaces of dilation and erosion are defined by

$$F_t f(x) = (f \oplus b_t)(x), \quad t > 0, \tag{12}$$
$$F_t f(x) = (f \ominus b_t)(x), \quad t > 0, \tag{13}$$

resp., with families of quadratic structure functions, $b_t = \|x\|^2/(2t)$. Here, dilation $\oplus$ and erosion $\ominus$ are given by

$$(f \oplus b)(x) = \max_{y \in \mathbb{R}^n} (f(x-y) - b(y)),$$
$$(f \ominus b)(x) = \min_{y \in \mathbb{R}^n} (f(x-y) + b(y)).$$

Motivated by the analogies between these scale spaces we look for a more general class of scale spaces on the ground of the algebraic operations introduced in section 2.1. We start with the definition of generalised dilations. Not all parameter values will lead to scale spaces in strict sense, i.e. with the semi-group property; we shall deal with this issue in proposition 2.

Definition 6. *Let $f : \mathbb{R}^n \to [0,1]$ be a signal and $b : \mathbb{R}^n \to \mathbb{R}_0^+$ continuous such that $\{x \in \mathbb{R}^n \mid b(x) < B\}$ is bounded for any $B \ge 0$. For $q \in [1, +\infty]$, $p \in [0,1]$ we define*

$$\left(f \oplus_{q,p} b\right)(x) := {}_q\!\!\int_{\mathbb{R}^n} f(x-y) -_p b(y)\,\mathrm{d}y \tag{14}$$

and call $f \oplus_{q,p} b$ the (q,p)-dilation of f w.r.t. the kernel b.

Obviously, ordinary dilation is recovered for $q = +\infty$, $p = 1$. For $q = 1$, $p = 0$, we have convolution with the kernel $1/b$. As a third special case we mention the "multiplicative dilation" $\oplus_{+\infty,0}$ with

$$\left(f \oplus_{+\infty,0} b\right)(x) = \sup_{y \in \mathbb{R}^n} f(x-y)/b(y).$$

It must be pointed out that our definition of $-_p$ implies that the range of $f(x-y)-_p b(y)$ is truncated from below at zero. Since f is assumed to have $[0,1]$ range, and $b_p(0)=0$, this has no effect whatsoever on the result of the generalised dilation as long as the integral is in fact a maximum, i.e. for $q=+\infty$. The truncation at zero has also no effect in the case $p=0$ since then the integrand is in fact $f(x-y)/b(y)$ which never becomes negative. However, for $q<+\infty$ and $p>0$ the truncation at zero is in fact somehow arbitrary; we come back to this issue in section 2.4 where the properties of the family of (q,p)-dilations will be discussed in more detail.

Proposition 1. *The (q,p)-dilation with kernel b is rotationally invariant for all continuous $f:\mathbb{R}^n\to\mathbb{R}_0^+$ if and only if $b(x)$ depends only on $\|x\|$. The (q,p)-dilation with kernel b is rotationally invariant and separable if and only if either $q=+\infty$, $p>0$, $b(x)=b_{p;\lambda}(x)=\lambda\,\|x\|^{2/p}$, or $p=0$, $q\in[1,+\infty]$ arbitrary, and $b(x)=b_{0;k,\sigma}(x)=k^n\exp(\|x\|^2/(2\sigma^2))$, with λ, k and σ being arbitrary positive real numbers.*

Proof. Rotational invariance means that whatever f may be given, $(f\oplus_{q,p}b)_\varrho$ is identical with $f_\varrho\oplus_{q,p}b$ for any rotation $\varrho:\mathbb{R}^n\to\mathbb{R}^n$. Here, g_ϱ is defined as $g_\varrho(x):=g(\varrho x)$ for all $x\in\mathbb{R}^n$. Now we have

$$\begin{aligned}(f\oplus_{q,p}b)_\varrho(x)&={}_q\!\int_{\mathbb{R}^n} f(\varrho x-\varrho y)+_p b(\varrho y)\,\mathrm{d}y\\&={}_q\!\int_{\mathbb{R}^n} f_\varrho(x-y)+_p b_\varrho(y)\,\mathrm{d}y\end{aligned}$$

on one side and

$$(f_\varrho\oplus_{q,p}b)(x)={}_q\!\int_{\mathbb{R}^n} f_\varrho(x-y)+_p b(y)\,\mathrm{d}y$$

on the other side. Identity for all f can hold only if $b_\varrho(x)=b(x)$ for all $x\in\mathbb{R}^n$ and all rotations ϱ which implies that $b(x)$ depends only on $\|x\|$ since rotations act transitive on each sphere $\|x\|=\text{const}$.

To study separability, it is sufficient to consider the decomposition of two-dimensional (q,p)-dilation into two one-dimensional (q,p)-dilations. Requiring that for all admissible functions f the two-dimensional (q,p)-dilation

$${}_q\!\int_{\mathbb{R}^2} f(x-y)+_p b(y)\,\mathrm{d}y,$$

$x=(x_1,x_2)^{\mathrm{T}}$, $y=(y_1,y_2)^{\mathrm{T}}$, be equal to the concatenation

$${}_q\!\int_{\mathbb{R}}{}_q\!\int_{\mathbb{R}} f((x_1-y_1,x_2-y_2)^{\mathrm{T}})+_p b_1(y_1)\,\mathrm{d}y_1+_p b_2(y_2)\,\mathrm{d}y_2$$

implies that the p-addition of $b(y_2)$ commutes with the inner q-integration, and $b_1(x_1)+_p b_2(x_2)=b(x)$ holds for the kernels b, b_1, b_2. The first restriction boils down to the distributivity of the two operations $+_q$, $+_p$ and thus to the condition ($q = +\infty$ or $p = 0$). Evaluation of the second condition for $p > 0$ together with rotational invariance leads to $b_i(x_i) = \lambda |x_i|^{2/p}$, $i = 1, 2$, and $b(x) = \lambda \|x\|^{2/p}$. For $p = 0$ the second condition becomes $b_1(x_1)b_2(x_2) = b(x)$; again, combination with rotational invariance yields $b_i(x_i) = k \exp(x_i^2/(2\sigma^2))$, $b(x) = k^2 \exp(\|x\|^2/(2\sigma^2))$ with constants k, σ.

Finally, one easily checks that with $q = +\infty$ or $p = 0$ and the described kernels one has indeed rotational invariance from the first part of the proposition and also the intended separability.

This result is in accordance with the known facts about dilation and convolution in cases $(q, p) = (1, 0), (+\infty, 1)$. Note that the exponent in b_0 has positive sign; this is just a side-effect of our choice of notation for the generalised dilation. The conventional Gauss kernel is $1/b_0$, with $k = \sigma\sqrt{2\pi}$.

With classical morphological operations, it can be observed that erosion and dilation are related via $1 - (f \ominus b) = (1 - f) \oplus b$ for all f, b. This allows us to introduce generalised erosion as follows.

Definition 7. *For $q \in [1, +\infty]$, $p \in [0, 1]$, $f : \mathbb{R}^n \to [0, 1]$ and $b : \mathbb{R}^n \to \mathbb{R}_0^+$ as in definition 6 let*

$$(f \ominus_{q,p} b)(x) := 1 - ((1 - f) \oplus_{q,p} b)(x). \tag{15}$$

We shall call this operation (q, p)-erosion.

From the above-mentioned relation between conventional dilation and erosion it is clear that ordinary erosion is recovered, again, for $(q, p) = (+\infty, 1)$. Note that the $(1, 0)$-erosion of f w.r.t. b is just the convolution of f and $1/b$, plus a constant which is zero for kernels of total weight 1.

Definition 8. *For each $(q, p) \in [1, +\infty] \times [0, 1]$ a family of dilation filters $F_t^{q,p} : \mathcal{F} \to \mathcal{F}$ can be defined by $F_0 := \mathrm{id}$ and, for $t > 0$, $F_t^{q,p} f := f \oplus b_{p;\lambda(t)}$ with $\lambda(t) \sim t^{-1/p}$ if $p > 0$, $F_t^{q,0} f := f \oplus b_{0;k,\sigma(t)}$ with $\sigma(t) \sim \sqrt{t}$. By replacing dilation with erosion in the definition of F_t, a family of erosion filters is obtained.*

These families of filters obey most properties of scale spaces – allow for rescaling to satisfy maximum-minimum principle – but still the question is open whether they are semi-groups. We answer this by the following proposition.

Proposition 2. *The (q, p)-dilation filters $F_t^{q,p}$, $t \geq 0$ from def. 8 form a semi-group if and only if $q = +\infty$ or $p = 0$. The same is true for (q, p)-erosion filters.*

Proof. By an easy computation it is seen that one has indeed $F_{t_2}^{q,p} \circ F_{t_1}^{q,p} = F_{t_1+t_2}^{q,p}$ if the condition on (q, p) is satisfied.

On the other hand, semi-group property requires that for given $t_1, t_2 > 0$ the concatenation $F_{t_2}^{q,p} \circ F_{t_1}^{q,p}$ can be represented by one single (q, p)-dilation $f \mapsto f \oplus_{q,p} b$ for all f, with b independent on f. Like in the proof of the separability statement of proposition 1 one concludes that to enable this, $+_q$ and $+_p$ have to fulfil a distributivity law. Thus, $q = +\infty$ or $p = 0$ is necessary. Transfer to erosions is obvious.

2.3 Limit statements

We want to investigate now in which sense the families of generalised morphological operations as introduced in defs. 6 and 7 of the previous section are continuous w.r.t. the parameters q and p. To this purpose, we assume that the input image f is arbitrarily chosen but fixed. Then it is clear that we have continuity – even uniform continuity – in q at any $(q,p) \in [1,+\infty) \times [0,1]$ and also in p at any $(q,p) \in [1,+\infty] \times (0,1]$ because of the continuity of the family of power functions used. It remains to describe the continuity of the transitions $q \to +\infty$, $p \to +0$. The following proposition deals with the limit in p. Note that as p tends to zero, the kernel parameters have to be adjusted.

Proposition 3. *Define $b_{0;k,\sigma}$ and $b_{p;\lambda}(x)$ for $p \in (0,1]$ as in proposition 1, with $\lambda = \lambda_p := (p/(2\sigma^2))^{1/p}$ for $p \in (0,1]$. Then we have for all $q \in [1,+\infty]$ and for any continuous, bounded function $f \in \mathcal{F}$, $f : \mathbb{R}^n \to \mathbb{R}_0^+$ the pointwise limit equations*

$$\lim_{p\to+0} \left(f \oplus_{q,p} b_{p;\lambda}\right)(x) = \left(f \oplus_{q,0} b_{0;1,\sigma}\right)(x), \tag{16}$$

$$\lim_{p\to+0} \left(f \ominus_{q,p} b_{p;\lambda}\right)(x) = \left(f \ominus_{q,0} b_{0;1,\sigma}\right)(x), \tag{17}$$

for all $x \in \mathbb{R}^n$.

Proof. First, we consider dilations with $q = +\infty$. We have

$$\begin{aligned}
\lim_{p\to+0} \left(f \oplus_{+\infty,p} b_{p;\lambda}\right)(x) &= \lim_{p\to+0} \max_y (f(x-y)^p - p\|y\|^2/(2\sigma^2))^{1/p} \\
&= \lim_{p\to+0} \max_y f(x-y)(1 - f(x-y)^{-p} p\|y\|^2/(2\sigma^2))^{1/p} \\
&= \max_y f(x-y) \lim_{p\to+0} (1 - f(x-y)^{-p} p\|y\|^2/(2\sigma^2))^{1/p} \\
&= \max_y f(x-y) \exp(-\|y\|^2/(2\sigma^2)) \\
&= \left(f \oplus_{+\infty,0} b_{0;1,\sigma}\right)(x).
\end{aligned}$$

Let now $q \in [1,+\infty)$. Then

$$\begin{aligned}
\lim_{p\to+0} \left(f \oplus_{q,p} b_{p;\lambda}\right)(x) &= \lim_{p\to+0} \left(\int_{D_{p,x}} (f(x-y)^p - p\|y\|^2/(2\sigma^2))^{1/p} \, \mathrm{d}y \right)^{1/q} \\
&= \left(\lim_{p\to+0} \int_{D_{p,x}} f(x-y)^q (1 - f(x-y)^{-p} p\|y\|^2/(2\sigma^2))^{q/p} \, \mathrm{d}y \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^n} f(x-y)^q \lim_{p\to+0} (1 - f(x-y)^{-p} p\|y\|^2/(2\sigma^2))^{q/p} \, \mathrm{d}y \right)^{1/q} \\
&= \left(\int_{\mathbb{R}^n} f(x-y)^q \exp(-q\|y\|^2/(2\sigma^2)) \, \mathrm{d}y \right)^{1/q} \\
&= \left(f \oplus_{q,0} b_{0;1,\sigma}\right)(x)
\end{aligned}$$

with $D_{p,x} := \{y \in \mathbb{R}^n \mid f(x-y) \geq b_{p;\lambda}(y)\}$, where we have made use of the monotonic convergence theorem.

Replacing f by $1-f$ and subtracting the resulting equations from 1, both limit results are easily transferred to erosions.

We turn now to the limit case $q \to +\infty$.

Proposition 4. *Let $p \in [0,1]$ fixed and $b : \mathbb{R}^n \to \mathbb{R}_0^+$ continuous and bounded. Then we have for any continuous, bounded $f \in \mathcal{F}$, $f : \mathbb{R}^n \to \mathbb{R}_0^+$ the pointwise limit equations*

$$\lim_{q \to +\infty} (f \oplus_{q,p} b)(x) = (f \oplus_{+\infty,p} b)(x), \tag{18}$$

$$\lim_{q \to +\infty} (f \ominus_{q,p} b)(x) = (f \ominus_{+\infty,p} b)(x). \tag{19}$$

Proof. For any f as required and $x \in \mathbb{R}^n$, one has

$$(f \oplus_{q,p} b)(x) = \left(\int_{D_x} g_x(y)^q \, \mathrm{d}y \right)^{1/q}$$

where $g_x(y) := (f(x-y)^p - b(y)^p)^{1/p}$ if $p > 0$, $g_x(y) = f(x-y)b(y)$ if $p = 0$, and $D_x = \{y \in \mathbb{R}^n \mid f(x-y) \geq b(y)\}$ if $p > 0$, $D_x = \mathbb{R}^n$ if $p = 0$. In both cases, $g_x(y)$ is continuous, bounded and takes only non-negative values on D_x. Thus,

$$\lim_{q \to +\infty} \left(\int_{D_x} g_x(y)^q \, \mathrm{d}y \right)^{1/q} = \sup_y g_x(y) = (f \oplus_{+\infty,p} b)(x).$$

Transfer to the erosion case is clear.

2.4 Properties of the (q,p)-dilations

We discuss now in more detail several features of the (q,p)-dilations. Everything said here transfers to (q,p)-erosions in an obvious way.

Let us first look at invariance properties of the (q,p)-dilations. Both morphological and Gaussian scale spaces are invariant under grey-value shifts, $f \mapsto f + C$. Gaussian scale space also displays invariance under scalar multiplication of grey-values, $f \mapsto C \cdot f$. We ask therefore which (q,p)-dilations share one of these invariances.

It turns out that the grey-value shift invariance is restricted to the two parameter pairs $(q,p) = (+\infty, 1), (1, 0)$ corresponding to morphological and Gaussian scale space themselves. Invariance under scalar multiplication of grey-values holds for all $(q,0)$-dilations. However, a closer look shows that the grey-shift invariance of ordinary dilation is not simply lost but turns into an invariance under the grey-value transform $f \mapsto f +_p C$ for $q = +\infty$, $p \in [0,1]$. In the limit $p = 0$ it coincides with the scalar multiplication invariance of the $(q,0)$ case. Thus, it is the grey-shift invariance in $(1,0)$ case which is truly an additional symmetry.

There is another way how the (q, p)-dilations for $q = +\infty$ or $p = 0$ can be understood. It coincides with one of the scaling operations used by Heijmans and van den Boomgaard in [9], making clear that these particular generalised dilations are also included in their framework. In $(+\infty, p)$-dilation, only the "inner" operation differs from that in ordinary dilation by the action of $\varphi : z \mapsto z^p$: simple addition is replaced by mapping the arguments via φ, executing the original addition and transforming back the result. In particular, the kernels b_p transform to simple quadratic kernels of the type b_1 under φ. Since φ is strictly increasing, the inverse mapping φ^{-1} commutes with taking the maximum. We can therefore describe $(+\infty, p)$-dilation as ordinary dilation performed on a signal which is obtained from the original one by a strictly monotonic transformation of grey-values, $f \oplus_{+\infty,p} b_p = \varphi^{-1}(\varphi(f) \oplus b_1)$ or, in terms of the filtering operators from def. 8, $F_t^{+\infty,p} = \varphi^{-1} \circ F_t^{+\infty,1} \circ \varphi$. An analogous argument applies to the $(q, 0)$-dilations with $q \in (1, +\infty)$. Again, the commutation of two operations is crucial – here, $\psi : z \mapsto z^q$ may be applied before, instead of after, the multiplication of $f(x - y)$ by $b_0(y)$, provided the Gaussian b_0 is replaced with $\tilde{b}_0 := b_0^q$ which is a Gaussian, too, just with different standard deviation. We have $f \oplus_{q,0} b_0 = \psi^{-1}(\psi(f) * (1/\tilde{b}_0))$ and $F_t^{q,0} = \psi^{-1} \circ F_{t'}^{1,0} \circ \psi$. Unfortunately, the grey-value transformation picture does not allow to include the case $(q, p) = (+\infty, 0)$ from either side.

As can be seen from the preceding paragraphs, (q, p)-dilations make sense for $(q, p) \in [1, +\infty] \times [0, 1]$. The algebraic definition is clear, and we have studied the continuity properties. However, there are considerable drawbacks for the parameter values $(q, p) \in Y := [1, +\infty) \times (0, 1]$ which strongly suggest that the boundary cases with $q = +\infty$ or $p = 0$ are actually the interesting ones.

First, we have pointed out earlier that the definition of the $-_p$ operation contains a truncation at zero which constitutes no problem for $q = +\infty$ or $p = 0$ since it does not influence the result. The truncation itself can't be avoided in this construction since the $+_p$ operations can't be defined for negative numbers in a sensible way. But for $(q, p) \in Y$ this truncation introduces an arbitrarity into the definition of (q, p)-dilations.

Second, since no distributivity law between $+_q$ and $+_p$ applies for $(q, p) \in Y$, it is not easy at all to interpret the algebraic operation of q-integrating over p-sums. Qualitatively, the image f and kernel b reduce their true interaction as q and p approach to each other, and for $q = p = 1$ the whole operation degenerates into a summation of paraboloid hats. An even more severe consequence of the lack of distributivity is, third, the non-separability for $(q, p) \in Y$. This constitutes an obstacle to efficient numerical computation of (q, p)-dilations with $(q, p) \in Y$.

Finally, the filter family $F_t^{q,p}$ with $(q, p) \in Y$ has no semi-group structure and is, therefore, not a scale space in strict sense.

2.5 Comparison to the construction of Florack et al.

We want now to compare our family of generalised morphological scale spaces to the family of pseudo-linear scale spaces introduced by Florack et al. in [8, 7] which also links morphological and Gaussian scale spaces.

Pseudo-linear scale spaces are introduced as a one-parameter deformation of linear Gaussian scale space via the grey-value transformation

$$\gamma_\mu : x \mapsto [x]_\mu := \frac{\exp(\mu x) - 1}{\exp \mu - 1}, \ \mu \in \mathbb{R} \setminus \{0\}, \quad \gamma_0 = \mathrm{id}. \tag{20}$$

Morphological dilation and erosion scale spaces with quadratic structure functions are recovered as limit cases $\mu \to \pm\infty$ of the pseudo-linear family.

As opposed to this, the approach presented here varies morphological scale spaces in a way that includes Gaussian scale space. More precisely, deformed versions of both scale space categories are given that share the $(+\infty, 0)$ limit case. While the algebraic operations used here are somewhat simpler and allow generalisations of the same type to be inserted for the "inner" and "outer" operations of the dilation, the proposal of Florack et al. has the clear advantage of being linked to a simple modification in the Laplace-Beltrami operator which is to be used when the pseudo-linear filter is to be described as a diffusion process – an aspect that could not be regarded in the present paper.

A crucial point in the limiting process on pseudo-linear scale spaces leading to the morphological scale spaces is that a rescaling of the standard deviation σ of the Gaussian kernel is used, such that $\sigma\sqrt{|\mu|}$ is kept constant while $|\mu|$ tends to infinity. It might be that this type of renormalisation during the transition process is principally inevitable – note that we had to use a quite analogous procedure in proposition 3.

3 Experiment

As an illustrating example we show the results of (q, p)-dilations with different values of q and p on a simple image showing a few geometrical figures contaminated with Gaussian noise. The original image is shown in fig. 1.

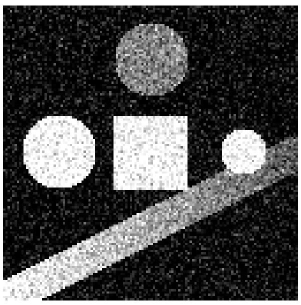

Fig. 1. The simple 128×128 image used to illustrate the parametrised dilations. A compilation of five simple geometric shapes is superposed by uncorrelated Gaussian noise with a standard deviation of 15 % of the highest grey value.

The dilated images (fig. 2) show the interpolation property of the family of (q, p)-dilations. Note how the granular structure typical for the ordinary dilation of noisy images at small t (left bottom) is gradually reduced as the parameters are changed towards those of ordinary Gaussian convolution (right top). Also, it is worth noting that those (q, p)-dilations having neither $q = +\infty$ nor $p = 0$, in spite of their theoretical shortcomings, do not turn out obviously disastrous in the numerical experiment. Of course, for lack of separability, they consume considerably more computing time.

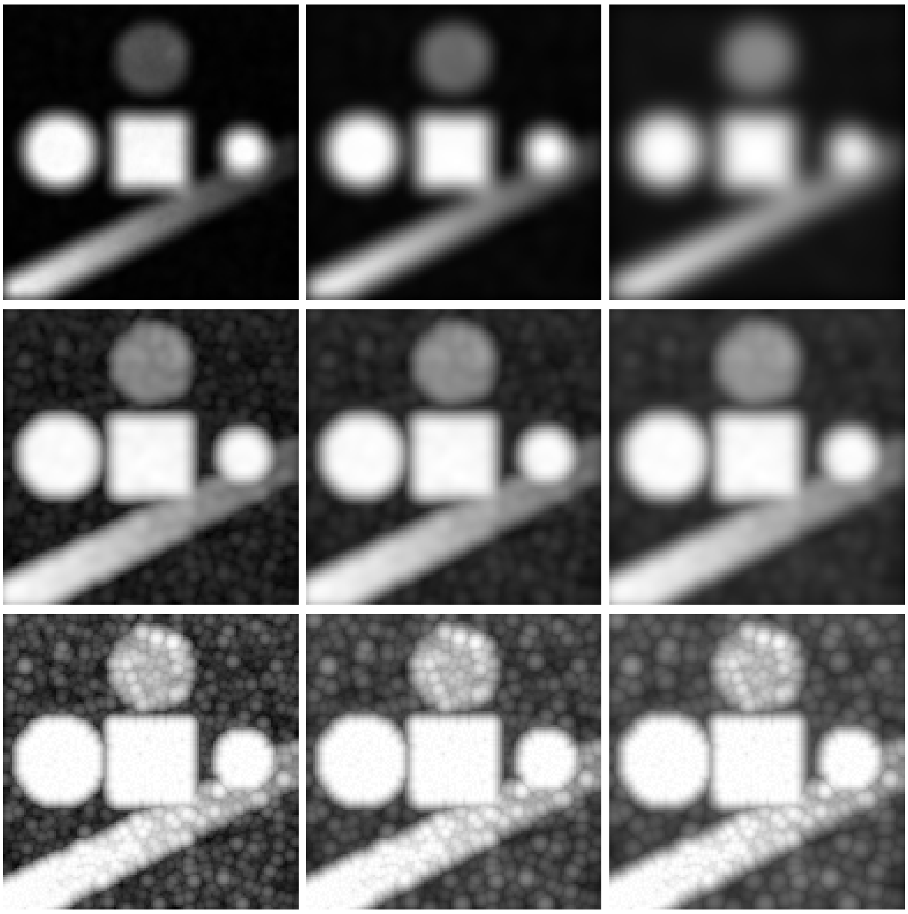

Fig. 2. Results of (q, p)-dilation of the simple image from fig. 1. Columns from left to right correspond to $p = 1, 0.5, 0$, rows from top to bottom correspond to $q = 1, 4, +\infty$. In all pictures, t^2 is set to 5. In the upper two rows the grey-values are linearly remapped to $[0, 1]$.

In fig. 3, the same dilated images are shown decorated with selected level-lines. The reduction of the granular structure becomes even more eye-catching, along with the changes in topology of the level-lines particularly in the transition zones between the geometrical elements. Finally, fig. 4 shows one-dimensional sections of the same images.

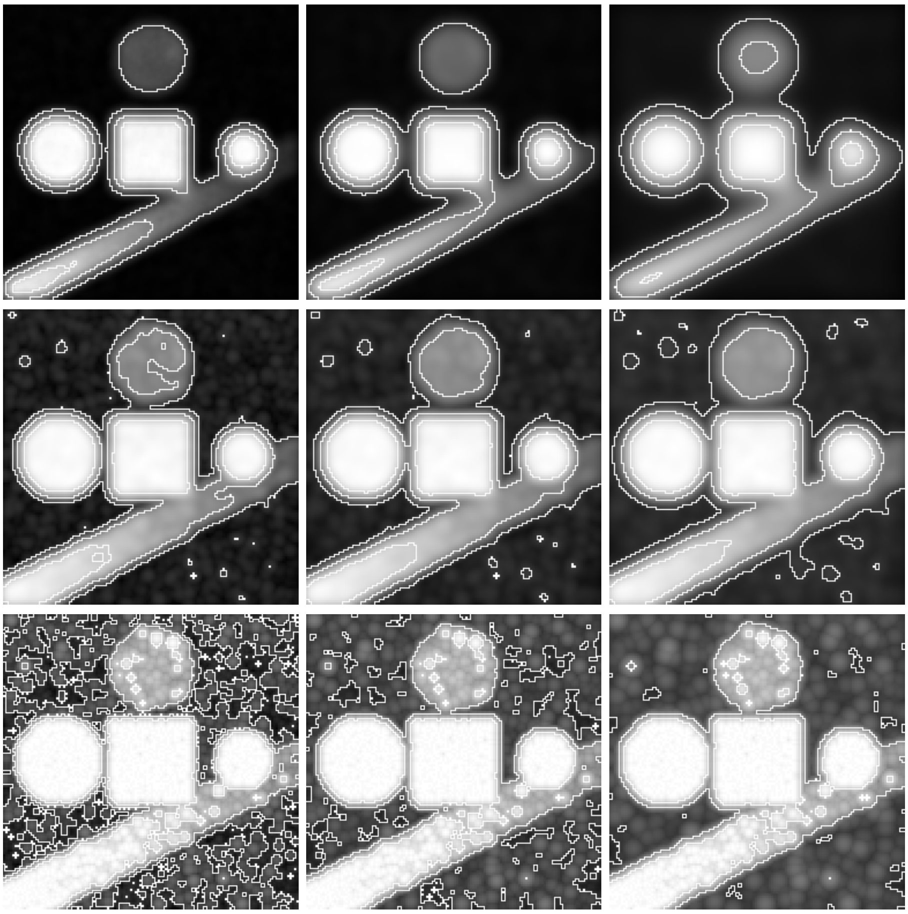

Fig. 3. The same dilated images as in fig. 2 but with level lines corresponding to 0.2, 0.5 and 0.8 times the highest grey-value.

4 Conclusion

We have introduced two-parameter families of generalised scale spaces that connect the well-studied morphological scale spaces of dilation and erosion with

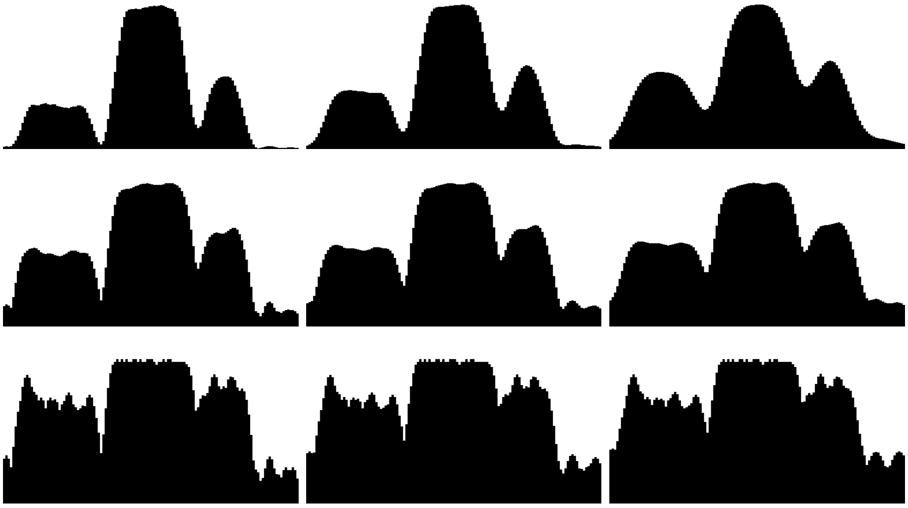

Fig. 4. One-dimensional sections of the images from fig. 2 along the vertical middle-axis.

the Gaussian convolution linear scale space. For distinguished sub-families, the semi-group property holds, making them into scale spaces in strict sense. The construction relies on a family of algebraic operations and integrals which correspond to l^p and L^p norms and generalised (power) means.

The results are primarily of theoretical interest in the theory of scale spaces since they hopefully will enrich the picture of structural analogies between the above-mentioned classes of scale spaces. In some sense, the construction presented here is complementary to that of pseudo-linear scale spaces by Florack et al. [8, 7]. In particular, both approaches share the need for a renormalisation of the scale parameter in the transition between morphological and Gaussian scale spaces.

An interesting point for possible applications is the simplicity of the algebraic operations used in defining the family of scale spaces which still allows for good control over their algebraic properties.

Future work should also deal with the question how the pseudo-linear scale space approach of Florack et al. and the construction shown here could be integrated into a unified framework. Investigations should include as well possible relations to diffusion-like processes. An extension of the construction to Fourier and slope transforms would be desirable.

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