DFG–Schwerpunktprogramm 1114

Mathematical methods for time series analysis and digital image processing

Splines in higher order TV regularization

Gabriele Steidl Stephan Didas
Julia Neumann

Preprint 142

Preprint Series DFG-SPP 1114

Preprint 142 June 2006

Splines in Higher Order TV Regularization

Gabriele Steidl [*] Stephan Didas [†] Julia Neumann [‡]

March 27, 2006

Abstract

Splines play an important role as solutions of various interpolation and approximation problems that minimize special functionals in some smoothness spaces. In this paper, we show in a strictly discrete setting that splines of degree $m-1$ solve also a minimization problem with quadratic data term and m-th order total variation (TV) regularization term. In contrast to problems with quadratic regularization terms involving m-th order derivatives, the spline knots are not known in advance but depend on the input data and the regularization parameter λ. More precisely, the spline knots are determined by the contact points of the m–th discrete antiderivative of the solution with the tube of width 2λ around the m–th discrete antiderivative of the input data. We point out that the dual formulation of our minimization problem can be considered as support vector regression problem in the discrete counterpart of the Sobolev space $W_{2,0}^m$. From this point of view, the solution of our minimization problem has a sparse representation in terms of discrete fundamental splines.

1 Introduction

In this paper, we are interested in the solution of the minimization problem

$$\frac{1}{2}\int_0^1 (u(x) - f(x))^2 + \lambda |u^{(m)}(x)| \,\mathrm{d}x \quad \to \quad \min \tag{1}$$

and some of its 2D versions involving first and second order partial derivatives. More precisely, we work in a strictly discrete setting which is appropriate for tasks in digital signal processing. For a discrete signal $\boldsymbol{u} =$

[*]steidl@math.uni-mannheim.de, University of Mannheim, Faculty of Mathematics and Computer Science , 68131 Mannheim, Germany

[†]didas@mia.uni-saarland.de, Mathematical Image Analysis Group, Faculty of Mathematics and Computer Science, Saarland University, 66123 Saarbrücken, Germany

[‡]jneumann@uni-mannheim.de, University of Mannheim, Faculty of Mathematics and Computer Science , 68131 Mannheim, Germany

$(u(1),\ldots,u(n))^{\mathrm{T}}$, we use the m-th *forward difference*

$$\triangle^m u(j) := \sum_{k=0}^{m}(-1)^{k+m}\binom{m}{k}u(j+k), \qquad j=1,\ldots,n-m \tag{2}$$

as discretization of the m-th derivative. Then, for given input data $\boldsymbol{f} \in \mathbb{R}^n$, we are looking for the solution of the minimization problem

$$\frac{1}{2}\sum_{j=1}^{n}(u(j)-f(j))^2 + \lambda \sum_{j=1}^{n-m} |\triangle^m u(j)| \quad \to \quad \min, \tag{3}$$

where we refer to the penalty term as m–order TV regularization. Of course, other discretizations of (1) are possible. In contrast to the solution of the well examined version of (3) with quadratic penalty term $|\triangle^m u(j)|^2$, the solution of (3) does not linearly depend on the input data. This results in some advantages over the linear solution as better edge preserving. For two dimensions and first order derivatives in the penalizer, problem (3) becomes the classical approach of Rudin, Osher and Fatemi (ROF) [23] which has many applications in digital image processing. Meanwhile there exist various solution methods for this problem, see [30] and the references therein. Most of these methods introduce a small additional smoothing parameter to cope with the non differentiability of $|\cdot|$. There are two approaches which avoid such an additional parameter, namely a wavelet inspired technique [32] and the Legendre–Fenchel dualization technique, see, e.g., [1, 4] which is also relevant in the present considerations. We further mention that other cost functionals than the quadratic one have to come into the play when dealing, e.g., with denoising of images corrupted with other than white Gaussian noise. In this context we only refer to recent papers of Nikolova et al. [21, 3] and the references therein.

In this paper, we are interested in the structure of the solution $\boldsymbol{u}$ even for $m > 1$. We show that $\boldsymbol{u}$ is a discrete spline of degree $m-1$, where the spline knots, in contrast to the linear problem with quadratic regularization term, depend on the input data $\boldsymbol{f}$ and on the regularization parameter λ. More precisely, the spline knots are determined by the contact points of the m–th discrete antiderivative of $\boldsymbol{u}$ with the tube of width 2λ around the m–th discrete antiderivative of $\boldsymbol{f}$. We will see that the dual formulation of our minimization problem can be considered as support vector regression (SVR) problem in the discrete counterpart of the Sobolev space $W_{2,0}^m$. The SVR problem can be solved by standard quadratic programming methods. This provides us with a sparse representation of $\boldsymbol{u}$ in terms of discrete fundamental splines. We formally extend the approach to two dimensions. Here further research has to be involved to see the relation, e.g., to classical radial basis functions.

This paper is organized as follows: since discrete approaches can be best described in matrix–vector notation, the next section introduces the basic difference operators as matrices. Section 3 shows that our minimization problem (3) is equivalent to a spline contact problem. To this end, we have to define discrete splines. Based on the dual formulation of our problem, Section 4 treats the spline contact problem as support vector regression problem and presents some denoising results. Section 5 gives future prospects to twodimensional problems. The paper is concluded with Section 6.

2 Difference Matrices

The discrete setting can be best handled using matrix-vector notation. To this end, we introduce the lower triangular $n \times n$ Toeplitz matrix

$$\boldsymbol{D}_n := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ -1 & 1 & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

By straightforward computation we see that the inverse of $\boldsymbol{D}_n$ is the addition matrix

$$\boldsymbol{A}_n := \boldsymbol{D}_n^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ 1 & 1 & \dots & 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}. \tag{4}$$

Remark 2.1 *While application of* $\boldsymbol{D}_n^m$ *is a discrete version of* m *times differentiation,* $\boldsymbol{A}_n^m$ *realizes* m*–fold integration, i.e.,* $\boldsymbol{A}_n^m \boldsymbol{f}$ *is a discrete version of the* m-th antiderivative of $\boldsymbol{f}$. *For example, the components of* $\boldsymbol{A}_n^m \boldsymbol{f}$ *are given for* $m = 1, 2$ *by*

$m = 1$	$m = 2$
$f(1)$	$f(1)$
$f(1) + f(2)$	$2f(1) + f(2)$
$f(1) + f(2) + f(3)$	$3f(1) + 2f(2) + f(3)$
$\vdots$	$\vdots$
$f(1) + f(2) + \ldots + f(n)$	$nf(1) + (n-1)f(2) + \ldots + f(n)$

and may be considered as discrete version of $A^1 f(x) = \int_0^x f(t)\,\mathrm{d}t$ *and* $A^2 f(x) = \int_0^x \int_0^{t_1} f(t)\,\mathrm{d}t\mathrm{d}t_1$*, respectively. For general* m*, the* j*–th component of* $\boldsymbol{A}_n^m \boldsymbol{f}$ *is* $\sum_{k=1}^{j} \frac{(j+1-k)^{(m-1)}}{(m-1)!} f(k)$*. Here* $k^{(m)} := 1$ *for* $m = 0$ *and* $k^{(m)} := k(k+1)\ldots(k+m-1)$ *for* $m \geq 1$ *is a discrete equivalent of the* m*–th power function.*

Let $\mathbf{0}_{n,m}$ denote the matrix consisting of $n \times m$ zeros, $\mathbf{1}_{n,m}$ the matrix consisting of $n \times m$ ones and $\boldsymbol{I}_n$ the $n \times n$ identity matrix. Then the m–th forward difference (2) can be realized by applying the m–th *forward difference matrix*

$$\boldsymbol{D}_{n,m} := (\mathbf{0}_{n-m,m}|\boldsymbol{I}_{n-m})\ \boldsymbol{D}_n^m$$

and our minimization problem (3) can be rewritten as

$$\frac{1}{2}\|\boldsymbol{f}-\boldsymbol{u}\|_2^2 + \lambda\|\boldsymbol{D}_{n,m}\boldsymbol{u}\|_1 \quad \to \quad \min. \tag{5}$$

The functional in (5) is strictly convex and has therefore a unique minimizer. The matrix $\boldsymbol{D}_{n,m}$ has full rank $n-m$, i.e., $\mathcal{R}(\boldsymbol{D}_{n,m}) = \mathbb{R}^{n-m}$. Moreover, the range $\mathcal{R}(\boldsymbol{D}_{n,m}^{\mathrm{T}})$ of $\boldsymbol{D}_{n,m}^{\mathrm{T}}$ and the kernel $\mathcal{N}(\boldsymbol{D}_{n,m})$ of $\boldsymbol{D}_{n,m}$ are given by

$$\begin{aligned}\mathcal{R}(\boldsymbol{D}_{n,m}^{\mathrm{T}}) &= \{\boldsymbol{f} \in \mathbb{R}^n : \sum_{j=1}^n j^r f(j) = 0,\ r = 0,\ldots,m-1\},\\ \mathcal{N}(\boldsymbol{D}_{n,m}) &= \operatorname{span}\{(j^r)_{j=1}^n : r = 0,\ldots,m-1\} \ = \ \Pi_{m-1},\end{aligned}$$

see, e.g., [7]. The space Π_m collects just the discrete polynomials of degree $\leq m$. Then we have the orthogonal decomposition

$$\boldsymbol{R}^n = \mathcal{R}(\boldsymbol{D}_{n,m}^{\mathrm{T}}) \oplus \mathcal{N}(\boldsymbol{D}_{n,m}). \tag{6}$$

Obviously, $\boldsymbol{D}_{n,m}$ is given by cutting of the first m rows of $\boldsymbol{D}_n^m$. The following relations between $\boldsymbol{D}_n^m$ and $\boldsymbol{D}_{n,m}$ are proved in the appendix.

Proposition 2.2 *The difference matrices fulfill the properties*

i) $\boldsymbol{D}_{n,m}^{T} = (-1)^m \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{I}_{n-m} \\ \mathbf{0}_{m,n-m} \end{pmatrix}$,

ii) $\boldsymbol{D}_{n,m}\boldsymbol{D}_n^m = \boldsymbol{D}_{n+m,2m} \begin{pmatrix} \mathbf{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix}$,

iii) $\boldsymbol{D}_{n+m,m} \begin{pmatrix} \mathbf{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} = \boldsymbol{D}_n^m$.

Proof.

i) Since $\boldsymbol{D}_{n,m}\boldsymbol{f} = (\Delta^m f(1),\ldots,\Delta^m f(n-m))^{\mathrm{T}}$ we can rewrite $\boldsymbol{D}_{n,m}$ as

$$\begin{aligned}\boldsymbol{D}_{m,n} &= \boldsymbol{D}_{n-(m-1),1} \cdot \ldots \cdot \boldsymbol{D}_{n,1}\\ &= (\mathbf{0}_{n-m,1}|\boldsymbol{I}_{n-m})\boldsymbol{D}_{n-(m-1)} \cdot \ldots \cdot (\mathbf{0}_{n-1,1}|\boldsymbol{I}_{n-1})\boldsymbol{D}_n\end{aligned}$$

Using that by definition

$$\boldsymbol{D}_{n,1}^{\mathrm{T}} = \boldsymbol{D}_n^{\mathrm{T}} \begin{pmatrix} \mathbf{0}_{1,n-1} \\ \boldsymbol{I}_{n-1} \end{pmatrix} = -\boldsymbol{D}_n \begin{pmatrix} \boldsymbol{I}_{n-1} \\ \mathbf{0}_{1,n-1} \end{pmatrix}$$

we obtain for the transposed matrix

$$\begin{aligned} \boldsymbol{D}_{n,m}^{\mathrm{T}} &= \boldsymbol{D}_{n,1}^{\mathrm{T}} \cdot \ldots \cdot \boldsymbol{D}_{n-(m-1),1}^{T} \\ &= (-1)^m \boldsymbol{D}_n \begin{pmatrix} \boldsymbol{I}_{n-1} \\ \boldsymbol{O}_{1,n-1} \end{pmatrix} \cdot \ldots \cdot \boldsymbol{D}_{n-(m-1),1} \begin{pmatrix} \boldsymbol{I}_{n-m} \\ \boldsymbol{0}_{1,n-m} \end{pmatrix}. \end{aligned}$$

Multiplication of $\boldsymbol{f}^T$ from the left is again successive application of first order differences. Equivalently we can apply m–th order finite differences and cut off all additional components which results in assertion i).

ii) By definition of $\boldsymbol{D}_{n,m}$ we have

$$\begin{aligned} \boldsymbol{D}_{n+m,2m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} &= (\boldsymbol{0}_{n-m,2m} | \boldsymbol{I}_{n-m})\ \boldsymbol{D}_{n+m}^{2m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} \\ &= (\boldsymbol{0}_{n-m,m} | \boldsymbol{I}_{n-m})\ (\boldsymbol{0}_{n,m} | \boldsymbol{I}_n)\ \boldsymbol{D}_{n+m}^{2m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix}. \end{aligned}$$

Since the cutoff of the first m rows and columns of a Toeplitz matrix results in the same Toeplitz matrix but with m times reduced order the last equation can be rewritten as

$$\boldsymbol{D}_{n+m,2m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} = (\boldsymbol{0}_{n-m,m} | \boldsymbol{I}_{n-m})\ \boldsymbol{D}_n^{2m}$$

and finally, by applying again the definition of $\boldsymbol{D}_{n,m}$ as

$$\boldsymbol{D}_{n+m,2m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} = \boldsymbol{D}_{n,m} \boldsymbol{D}_n^m.$$

iii) Using the definition of $\boldsymbol{D}_{n,m}$, we obtain

$$\boldsymbol{D}_{n+m,m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} = (\boldsymbol{0}_{n,m} | \boldsymbol{I}_n)\ \boldsymbol{D}_{m+n}^{m} \begin{pmatrix} \boldsymbol{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} = \boldsymbol{D}_n^m.$$

This completes the proof. □

3 Spline Contact Problem

In this section, we will see that our higher order TV problem (5) is equivalent to a discrete spline interpolation problem, where the spline knots are not known in advance but depend on the input data $\boldsymbol{f}$ and λ. For $m = 1$, the resulting spline contact problem is well examined and can be solved by the so–called 'taut string algorithm', see, e.g., [10].

A necessary and sufficient condition for $\boldsymbol{u}$ to be the minimizer of (5) is that the zero vector is an element of the functional's subgradient

$$\mathbf{0}_{n,1} \in \boldsymbol{u} - \boldsymbol{f} + \lambda\,\partial\|\boldsymbol{D}_{n,m}\boldsymbol{u}\|_1\,.$$

By [22, Theorem 23.9] and since the subgradient of $|x|$ is given by

$$\frac{x}{|x|} := \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ [-1,1] & \text{if } x = 0, \end{cases}$$

this can be rewritten as

$$\boldsymbol{u} \in \boldsymbol{f} - \lambda\,\boldsymbol{D}_{n,m}^{\mathrm{T}} \frac{\boldsymbol{D}_{n,m}\boldsymbol{u}}{|\boldsymbol{D}_{n,m}\boldsymbol{u}|},$$

where $\cdot/|\cdot|$ is taken componentwise. These inclusions in their present form are not very convenient for the computation of $\boldsymbol{u}$. However, multiplying with $\boldsymbol{A}_n^m$ and applying Proposition 2.2i) leads to

$$\boldsymbol{A}_n^m\boldsymbol{u} \in \boldsymbol{A}_n^m\boldsymbol{f} - (-1)^m\,\lambda \begin{pmatrix} \boldsymbol{I}_{n-m} \\ \mathbf{0}_{m,n-m} \end{pmatrix} \frac{\boldsymbol{D}_{n,m}\boldsymbol{u}}{|\boldsymbol{D}_{n,m}\boldsymbol{u}|}.$$

Setting

$$\begin{pmatrix} \boldsymbol{F}_I \\ \boldsymbol{F}_R \end{pmatrix} := \boldsymbol{A}_n^m\boldsymbol{f}, \quad \begin{pmatrix} \boldsymbol{U}_I \\ \boldsymbol{U}_R \end{pmatrix} := \boldsymbol{A}_n^m\boldsymbol{u} \tag{7}$$

with the splitting into the inner vector $\boldsymbol{F}_I \in \mathbb{R}^{n-m}$ and the right boundary vector $\boldsymbol{F}_R \in \mathbb{R}^m$, the inclusions can be rewritten as

$$\begin{aligned} \boldsymbol{U}_I &\in \boldsymbol{F}_I - (-1)^m\,\lambda\,\frac{\boldsymbol{D}_{n,m}\boldsymbol{u}}{|\boldsymbol{D}_{n,m}\boldsymbol{u}|}, \\ \boldsymbol{U}_R &= \boldsymbol{F}_R. \end{aligned}$$

It remains to replace $\boldsymbol{D}_{n,m}\boldsymbol{u}$. By (7) and (4), we see that

$$\boldsymbol{f} = \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{F}_I \\ \boldsymbol{F}_R \end{pmatrix}, \quad \boldsymbol{u} = \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{U}_I \\ \boldsymbol{U}_R \end{pmatrix} \tag{8}$$

and further by Proposition 2.2ii) that

$$\boldsymbol{D}_{n,m}\boldsymbol{u} = \boldsymbol{D}_{n+m,2m} \begin{pmatrix} \mathbf{0}_{m,n} \\ \boldsymbol{I}_n \end{pmatrix} \begin{pmatrix} \boldsymbol{U}_I \\ \boldsymbol{U}_R \end{pmatrix}.$$

Introducing an artificial left boundary $\boldsymbol{U}_L := \mathbf{0}_{m,1}$ and extending our vector by

$$\boldsymbol{U} := (\boldsymbol{U}_L^{\mathrm{T}}, \boldsymbol{U}_I^{\mathrm{T}}, \boldsymbol{U}_R^{\mathrm{T}})^{\mathrm{T}}$$

our inclusions become finally

$$\begin{aligned} \boldsymbol{U}_I &\in \boldsymbol{F}_I - (-1)^m \lambda \frac{\boldsymbol{D}_{n+m,2m}\boldsymbol{U}}{|\boldsymbol{D}_{n+m,2m}\boldsymbol{U}|}, \\ \boldsymbol{U}_R &= \boldsymbol{F}_R. \end{aligned}$$

Consequently, $\boldsymbol{U}$ is the unique solution of the following spline contact problem, where we have to explain the spline notation later.

Spline Contact Problem

(C1) Boundary conditions: $\boldsymbol{U}_L = \boldsymbol{0}_{m,1}$ and $\boldsymbol{U}_R = \boldsymbol{F}_R$.

(C2) Tube condition: $\|\boldsymbol{F}_I - \boldsymbol{U}_I\|_\infty \le \lambda$
$\boldsymbol{U}_I$ lies in a tube around $\boldsymbol{F}_I$ of width 2λ.

(C3) Contact condition:
Let $\Lambda_I := \{j \in \{m+1, \dots, n-m\} : \triangle^{2m} U(j-m) \neq 0\}$.
If $j \in \Lambda_I$, then $U(j)$ contacts the boundary of the tube, where
$(-1)^m \triangle^{2m} U(j-m) > 0 \Longrightarrow U(j) = F(j) - \lambda$ (lower contact),
$(-1)^m \triangle^{2m} U(j-m) < 0 \Longrightarrow U(j) = F(j) + \lambda$ (upper contact).

Remark 3.1 (Continuous and Discrete Natural Splines)
We recall that a natural polynomial spline of degree $2m-1$ with knots $x_1 < \ldots < x_r$ *is a function* $s \in C^{2m-2}$ *such that*

$$\begin{aligned} s^{(2m)}(x) &= 0, \quad \text{for } x \in (x_j, x_{j+1}),\ j = 1, \dots, r-1, \\ s^{(m)}(x) &= 0, \quad \text{for } x < x_1,\ x > x_r. \end{aligned}$$

These splines are the solutions in W^m*, the Sobolev space of* $(m-1)$ *times continuousely differentialble functions with* m*-th weak derivative in* L_2*, of*

$$\begin{aligned} &\frac{1}{2}\|f^{(m)}\|_2^2 \quad \to \quad \min \\ &\text{s.t.} \quad f(x_j) = \gamma_j, \quad j = 1, \dots, r. \end{aligned}$$

Mangasarian and Schumaker [17, 18] *have introduced the* discrete natural polynomial spline of degree $2m-1$ with knots $\Xi = \{i_1, \dots, i_r\}$, $i_j < i_k$ *for* $j < k$*, as a vector* $\boldsymbol{s} = (s(1), \dots, s(N))^T$ *which satisfies for* $j \notin \Xi$ *the relations*

$$\begin{aligned} \triangle^{2m} s(j-m) &= 0, \quad j = m+1, \dots, N-m; \\ \triangle^m s(j) &= 0, \quad j = 1, \dots, i_1 - 1;\ i_r + 1, \dots, N-m. \end{aligned}$$

As its continuous analogue the discrete natural polynomial spline of degree $2m-1$ *solves the minimization problem*

$$\begin{aligned} &\frac{1}{2}\sum_{j=1}^{N-m} (\triangle^m y(j))^2 \quad \to \quad \min \qquad (9) \\ &\text{s.t.} \quad y(i_j) = \gamma_j, \quad j = 1, \dots, r. \end{aligned}$$

For relations between continuous and natural spline in the limiting process $N \to \infty$ see also [17, 18].

Setting $N := n + m$ and using the spline knots $\Xi = \{1, \ldots, m\} \cup \Lambda_I \cup \{n - m + 1, \ldots, n\}$, we can interpret $\boldsymbol{U}$ defined by (C1) - (C3) is a discrete natural polynomial spline of degree $2m - 1$. In contrast to (9), the inner spline knots Λ_I are only determined by (C3) and not known in advance. This reflects the nonlinear character of our problem solution.

We extend the discrete spline concept to splines of even degree as follows: we call $\boldsymbol{s} = (s(1), \ldots, s(n))^{\mathrm{T}}$ a *discrete spline of degree* $m-1$ *with inner knots* $\Xi = \{i_1, \ldots, i_r\} \subseteq \{\lfloor \frac{m}{2} \rfloor + 1, \ldots, n - \lfloor \frac{m+1}{2} \rfloor\}$ if

$$\triangle^m s(j - \lfloor \frac{m}{2} \rfloor) = 0, \quad j = \lfloor \frac{m}{2} \rfloor + 1, \ldots, n - \lfloor \frac{m+1}{2} \rfloor; \; j \notin \Xi.$$

Then the discrete interpolation problem

$$s(i_j) = \gamma_j, \quad i_j \in \Xi \cup \{1, \ldots, \lfloor \frac{m}{2} \rfloor\} \cup \{n - \lfloor \frac{m+1}{2} \rfloor + 1, \ldots, n\}$$

has a unique solution. Thus, for given spline knots Λ_I, we could solve a spline interpolation problem. Unfortunately, the spline knots depend on the input data $\boldsymbol{f}$ and λ. Therefore, the solution of the spline contact problem in its present form is only convenient for $m = 1$, see Remark 3.2. For larger m and the continuous setting, an attempt to solve the contact problem is contained in [16]. For our discrete setting, we will see in the following section that the contact problem can be treated by simply solving a constraint quadratic minimization problem.

Remark 3.2 (Taut String Algorithm for $m = 1$)
For $m = 1$, condition (C3) *means that the polygon through $\boldsymbol{U}$ is convex at upper contact points and concave at lower contact points. Thus, the construction of $\boldsymbol{U}$ satisfying* (C1) – (C3) *is equivalent to the construction of the uniquely determined taut string within the tube around F of width 2λ fixed at $(0, 0)$ and $(n, F(n))$. In other words, the polygon through $\boldsymbol{U}$ has minimal lengths within the tube, i.e., it minimizes*

$$\sum_{j=0}^{n-1} \left(1 + (U(j+1) - U(j))^2\right)^{1/2},$$

subject to the tube and boundary conditions. An example of a taut string is shown in Figure 1. *For solving this problem there exists a very efficient algorithm of complexity $\mathcal{O}(n)$, the so–called* 'taut string algorithm'*, which is based on a convex hull algorithm, see, e.g.,* [6, 16].

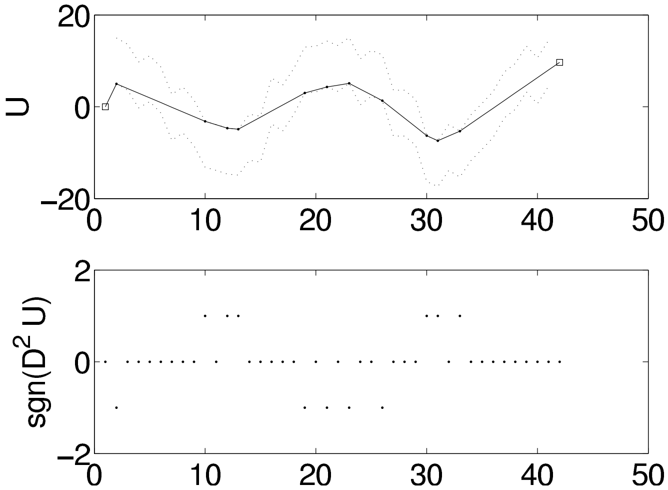

Figure 1: Solution of the spline contact problem (C1) – (C3) for a signal $\boldsymbol{F}$ of lengths $n+m$ with $n=40$ and $m=1$.

Interestingly, it was shown in [27, 33] that for $m=1$ the spline knots fulfill a so–called '*tree–property*'.

Remark 3.3 (Tree Property of Spline Knots for $m=1$)
Let $\lambda_{\max}$ be the smallest regularization parameter such that $\Lambda_I = \emptyset$. It is not hard to show that $\lambda_{\max} = \|P\boldsymbol{f}\|_{W_1(\boldsymbol{D}_{n,1})'}$, where P denotes the orthogonal projection of $\boldsymbol{f}$ onto $\mathcal{R}(\boldsymbol{D}_{n,1}^T)$ and $W_1(\boldsymbol{D}_{n,1})'$ is the dual space of $W_1(\boldsymbol{D}_{n,1}) := \mathcal{R}(\boldsymbol{D}_{n,1}^T)$ equipped with the norm $\|\boldsymbol{u}\|_{W_1(\boldsymbol{D}_{n,1})} := \|\boldsymbol{D}_{n,1}\boldsymbol{u}\|_1$.

If λ moves from $\lambda_{\max}$ to 0 and $\Lambda_I(\lambda)$ denotes the corresponding set of inner spline knots, then, for $\lambda_j > \lambda_k$,

$$\emptyset = \Lambda_I(\lambda_{\max}) \subseteq \Lambda_I(\lambda_j) \subseteq \Lambda_I(\lambda_k) \subseteq \Lambda_I(0) = \{m+1,\dots,n-m\}.$$

Figure 2 shows a tree of inner spline knots. The tree property does not hold for $m \geq 2$.

4 Support Vector Regression with Spline Kernels

In this section we want to show the relation of the discrete spline contact problem with discrete SVR. We start by a brief introduction to SVR in the continuous setting, where we emphasize the role of splines in the solution of the SVR problem in Sobolev spaces. Then we switch to the discrete context to explain the solution of (5) from the SVR point of view.

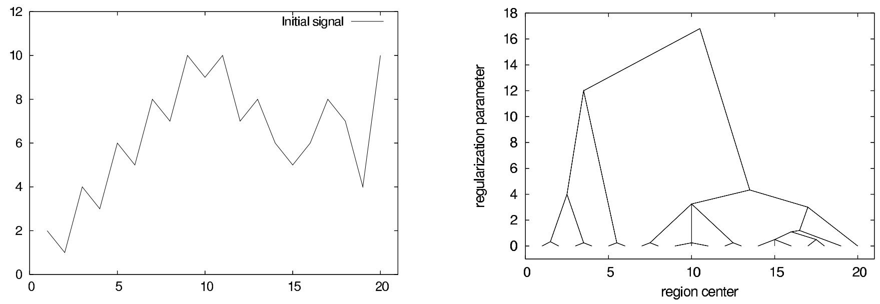

Figure 2: Original signal $\boldsymbol{f}$ (left), tree of spline knots with increasing regularization parameter λ from leaves to root (right).

4.1 Support Vector Regression - Continuous Approach

The SVR method searches for approximations of functions in reproducing kernel Hilbert spaces (RKHS) and plays an important role, e.g., in Learning Theory. Among the large amount of literature on SVR we refer to [29, Chapter 11]. SVR can be briefly explained as follows: Let $H \subset L_2(\mathbb{R}^d)$ be a Hilbert space with inner product $(\cdot,\cdot)_H$ having the property that the point evaluation functional is continuous. Then H possesses a so–called reproducing kernel $K \in L_2(\mathbb{R}^d \times \mathbb{R}^d)$ with reproducing property $(F, K(\cdot, x_j))_H = F(x_j)$ for all $F \in H$ and is called a *reproducing kernel Hilbert space* (RKHS). Given some function values $F(x_j)$, $j = 1, \ldots, p$, the *soft margin SVR* problem consists in finding a function $U \in H$ which minimizes

$$\mu \sum_{j=1}^{p} V_\lambda(F(x_j) - U(x_j)) + \frac{1}{2}\|U\|_H^2,$$

where $V_\lambda(x) := \max\{0, |x| - \lambda\}$ denotes Vapnik's λ-insensitive loss function. In other words, Vapnik's cost functional penalizes those $U(x_j)$ lying not in a λ neighbourhood of $F(x_j)$. If μ tends to infinity, then our cost functional must become zero and we obtain the *hard margin SVR* problem

$$\begin{aligned} &\frac{1}{2}\|U\|_H^2 \quad \to \quad \min \\ \text{s.t.} \quad &|F(x_j) - U(x_j)|_\infty \le \lambda, \quad j = 1, \ldots, p. \end{aligned} \tag{10}$$

By the Representer Theorem of Kimmeldorf and Wahba [14], the solution of (10) has the form

$$U(x) = \sum_{k=1}^{p} c(k) K(x_k, x),$$

i.e., only the given knots x_k are involved into the representation. Then (10) can be rewritten as

$$\begin{aligned} \frac{1}{2}\boldsymbol{c}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{c} \quad &\to \quad \min \\ \text{s.t.} \quad \|\boldsymbol{F}-\boldsymbol{K}\boldsymbol{c}\|_\infty &\leq \lambda \end{aligned} \tag{11}$$

with $\boldsymbol{F} := (F(x_j))_{j=1}^p$, $\boldsymbol{c} := (c(k))_{k=1}^p$ and $\boldsymbol{K} := (K(x_j,x_k))_{j,k=1}^p$. This is the usual hard margin SVR formulation.

Based on the Karush – Kuhn – Tucker conditions it follows that $c(k) \neq 0$ implies $|F(x_k) - U(x_k)| = \lambda$. Let

$$\Lambda := \{k \in \{1,,\ldots,p\} : c(k) \neq 0\}.$$

Then the solution U can be rewritten as

$$U(x) = \sum_{k\in\Lambda} c(k)K(x_k,x). \tag{12}$$

The functions $K(x_k,x)$ with $k \in \Lambda$ are called *support vectors*. Obviousely, U depends only on these support vectors and has a sparse representation in terms of the support vectors if $|\Lambda|$ is small compared to p. In the image processing context, SVR regression is mainly applied in high dimensional function spaces ($d \gg 1$), where often the Gaussian is involved as reproducing kernel.

For our purposes we will consider other well–known reproducing kernel Hilbert spaces, namely the Sobolev spaces $H = W_{2,0}^m$ of real–valued functions on $\mathbb{R}$ having a weak m–th derivative in $L_2\,[0,1]$ and fulfilling $F^{(r)}(0) = 0$ for $r = 0,\ldots,m-1$ with inner product

$$\langle F, G\rangle_{W_{2,0}^m} := \int_0^1 F^{(m)}(x)G^{(m)}(x)\,\mathrm{d}x.$$

These RKHS were for example considered in [31, p. 5–14]. The reproducing kernel in $W_{2,0}^m$ is

$$K(x,y) := \int_0^1 (x-t)_+^{m-1}\,(y-t)_+^{m-1}/((m-1)!)^2\,\mathrm{d}t, \tag{13}$$

where $(x)_+ := \max\{0,x\}$. For fixed y, the functions $K(\cdot,y)$ are splines fulfilling $K(\cdot,y) \in C^{2m-2}$, $K(\cdot,y) \in \Pi_{2m-1}$ in $[0,y]$ and $K(\cdot,y) \in \Pi_{m-1}$ in $[y,1]$.

In this context we mention that another minimization problem having so–called smoothing splines as solutions was considered the literature, see, e.g., [31, 28]: find $U \in W_{2,0}^m$ such that

$$\frac{1}{2}\sum_{j=1}^{p}(F(x_j) - U(x_j))^2 + \lambda\|U\|_{W_{2,0}^m}^2 \quad \to \quad \min$$

Again by the Representer Theorem, this problem has a solution of the form $U = \sum_{k=1}^{p} c(k)\, K(\cdot, x_k)$. Consequently, U is a continuous spline of degree $2m-1$ with knots x_k, $k = 1, \ldots, p$. However, in contrast to the solution (12) of (10), all coefficients $c(k)$ are in general $\neq 0$ and we obtain no sparse representation.

4.2 Support Vector Regression - Discrete Approach

To see the relation between our spline contact problem and SVR methods, we consider the dual formulation of problem (5).

Proposition 4.1 *The solution* $\boldsymbol{u}$ *of* (5) *is given by* $\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{D}_{n,m}^T \boldsymbol{V}_I$*, where* $\boldsymbol{V}_I$ *is the unique solution of the minimization problem*

$$\begin{aligned} & \frac{1}{2}\, \|\boldsymbol{f} - \boldsymbol{D}_{n,m}^T \boldsymbol{V}_I\|_2^2 \quad \to \quad \min \\ \text{s.t.} \quad & \|\boldsymbol{V}_I\|_\infty \leq \lambda. \end{aligned} \tag{14}$$

For a proof see, e.g., [25].

By (8) and Proposition 2.2 i) and iii) we obtain that

$$\begin{aligned} \|\boldsymbol{f} - \boldsymbol{D}_{n,m}^{\mathrm{T}} \boldsymbol{V}_I\|_2 &= \|\boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{F}_I \\ \boldsymbol{F}_R \end{pmatrix} - (-1)^m \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{I}_{n-m} \\ \boldsymbol{0}_{m,n-m} \end{pmatrix} \boldsymbol{V}_I\|_2 \\ &= \|\boldsymbol{D}_{n+m,m}(\boldsymbol{F} - (-1)^m \boldsymbol{V})\|_2, \end{aligned}$$

where $\boldsymbol{V} := (\boldsymbol{0}_{m,1}^{\mathrm{T}}, \boldsymbol{V}_I^{\mathrm{T}}, \boldsymbol{0}_{m,1}^{\mathrm{T}})^{\mathrm{T}}$. Setting $\boldsymbol{U} := \boldsymbol{F} - (-1)^m \boldsymbol{V}$, problem (14) can be rewritten as

$$\begin{aligned} & \frac{1}{2}\|\boldsymbol{D}_{n+m,m}\boldsymbol{U}\|_2^2 \quad \to \quad \min \\ \text{s.t.} \quad & \|\boldsymbol{F}_I - \boldsymbol{U}_I\|_\infty \leq \lambda, \quad \boldsymbol{U}_R = \boldsymbol{F}_R. \end{aligned} \tag{15}$$

The unique solution $\boldsymbol{U}$ of this problem which can be computed by standard quadratic programming (QP) methods is also the unique solution of our spline contact problem. Figure 3 illustrates the solution for $m = 3$.

Remark 4.2 *Regarding Remark* 3.2*, we see that for* $m = 1$ *the minimization problems*

$$\sum_{j=1}^{n} \left(1 + (U(j+1) - U(j))^2\right)^{1/2} \quad \to \quad \min,$$

and

$$\|\boldsymbol{D}_{n+1,1}\boldsymbol{U}\|_2^2 = \sum_{j=1}^{n} (U(j+1) - U(j))^2 \quad \to \quad \min$$

subject to the tube and boundary constraints lead to the same solution.

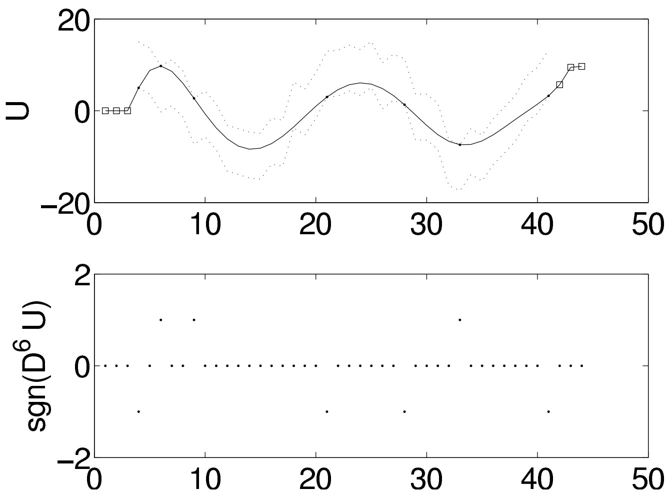

Figure 3: Solution of the spline contact problem (C1) – (C3) for a signal $\boldsymbol{F}$ of lengths $n+m$ with $n=40$ and $m=3$.

We will see that problem (15) can be considered as a *hard margin SVR problem*. To this end, we only have to define the appropriate RKHS. Let $\mathcal{W}_{2,0}^m := \{\boldsymbol{F} \in \mathbb{R}^{n+m} : F(j) = 0,\ j = 1, \ldots, m\}$ equipped with the inner product

$$\begin{aligned} \langle \boldsymbol{F}, \boldsymbol{G} \rangle_{\mathcal{W}_{2,0}^m} &:= \sum_{j=1}^{n} \triangle^m F(j) \triangle^m G(j) \\ &:= \langle \boldsymbol{D}_{n+m,m} \boldsymbol{F}, \boldsymbol{D}_{n+m,m} \boldsymbol{G} \rangle \\ &= \left\langle \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{F}_I \\ \boldsymbol{F}_R \end{pmatrix}, \boldsymbol{D}_n^m \begin{pmatrix} \boldsymbol{G}_I \\ \boldsymbol{G}_R \end{pmatrix} \right\rangle. \end{aligned}$$

Then the minimization term in (15) is just the norm of $\boldsymbol{U}$ in $\mathcal{W}_{2,0}^m$. Now we can straightforwardly determine the reproducing kernel in $\mathcal{W}_{2,0}^m$. Setting

$$\boldsymbol{K} := ((\boldsymbol{D}_n^m)^{\mathrm{T}} \boldsymbol{D}_n^m)^{-1} = \boldsymbol{A}_n^m (\boldsymbol{A}_n^m)^{\mathrm{T}}, \tag{16}$$

we see that the columns $\boldsymbol{K}_{0,k}$ of

$$\boldsymbol{K}_0 := (\boldsymbol{0}_{n,m} | \boldsymbol{K})^{\mathrm{T}} \in \mathbb{R}^{n+m,n}$$

form a special basis of $\mathcal{W}_{2,0}^m$, namely with reproducing property $\langle \boldsymbol{F}, \boldsymbol{K}_{0,j} \rangle_{\mathcal{W}_{2,0}^m} = F(j+m)$. Let us have a closer look at the structure of $\boldsymbol{K}$. Straightforward computation shows that the components of our discrete kernel are given by the discrete counterpart of (13), namely

$$K(j,k) = \sum_{r=0}^{\min(j,k)-1} (j-r)^{(m-1)} \, (k-r)^{(m-1)} / ((m-1)!)^2,$$

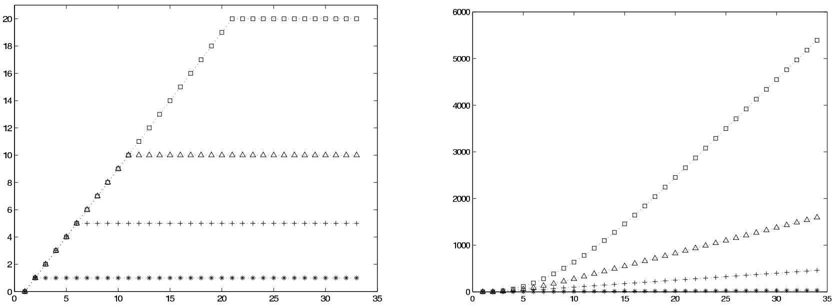

Figure 4: Discrete splines $\boldsymbol{K}_{0,k}$, $k = 1, 5, 10, 20$, for $n = 32$ and $m = 1$ (left), $m = 2$ (right).

with (m) defined as in Remark 2.1. By Proposition 2.2 ii) and i) we obtain that

$$\begin{aligned}\boldsymbol{D}_{n+m,2m}\boldsymbol{K}_0 &= \boldsymbol{D}_{n+m,2m}\begin{pmatrix}\boldsymbol{0}_{m,n}\\ \boldsymbol{I}_n\end{pmatrix}\boldsymbol{A}_n^m(\boldsymbol{A}_n^m)^{\mathrm{T}} = \boldsymbol{D}_{n,m}\boldsymbol{D}_n^m\boldsymbol{A}_n^m(\boldsymbol{A}_n^m)^{\mathrm{T}}\\ &= (-1)^m\left(\boldsymbol{I}_{n-m}, \boldsymbol{0}_{n-m,m}\right).\end{aligned}$$

In other words, we have for $j = m+1, \ldots, n-m$ that

$$\begin{aligned}\triangle^{2m}K_{0,k}(j-m) &= 0, \quad k = 1, \ldots, n-m;\ j \neq k,\\ \triangle^{2m}K_{0,k}(k-m) &= (-1)^m, \quad k = 1, \ldots, n-m,\\ \triangle^{2m}K_{0,k}(j-m) &= 0, \quad k = n-m+1, \ldots, n,\end{aligned}\tag{17}$$

i.e., $\boldsymbol{K}_{0,k}$ is a discrete spline of degree $2m-1$ with one inner knot $k+m$ for $k = 1, \ldots, n-m$ and a discrete polynomial in Π_{2m-1} for $k = n-m+1, \ldots, n$. For $n = 32$ and $m = 1, 2$, some columns of $\boldsymbol{K}_0$ are depicted in Figure 4.

For every $\boldsymbol{U} \in \mathcal{W}_{2,0}^m$, there exists a uniquely determined $\boldsymbol{c} \in \mathbb{R}^n$ such that $\boldsymbol{U} = \boldsymbol{K}_0\boldsymbol{c}$ and by the reproducing property of $\boldsymbol{K}_0$, problem (15) can be rewritten as

$$\begin{aligned}\frac{1}{2}\,\boldsymbol{c}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{c} \quad &\to \quad \min\\ \text{s.t.} \quad \|\boldsymbol{F}_I - (\boldsymbol{K}\boldsymbol{c})_I\|_\infty \leq \lambda, &\quad (\boldsymbol{K}\boldsymbol{c})_R = \boldsymbol{F}_R.\end{aligned}\tag{18}$$

This is the usual form (11) of a hard margin SVR problem. Let $\boldsymbol{c}$ be the solution of (18) and let

$$\tilde{\Lambda}_I := \{j \in \{m+1, \ldots, n\} : c(j-m) \neq 0\}$$

so that

$$\boldsymbol{U} = \sum_{j \in \tilde{\Lambda}_I} c(j-m)\boldsymbol{K}_{0,j-m} + \sum_{j=n-m+1}^{n} c(j)\boldsymbol{K}_{0,j}. \tag{19}$$

The vectors $\boldsymbol{K}_{0,j-m}$, $j \in \tilde{\Lambda}_I$ are called (inner) *support vectors*. By (19) and property (17) of $\boldsymbol{K}_0$ they are related to the spline knots as follows:

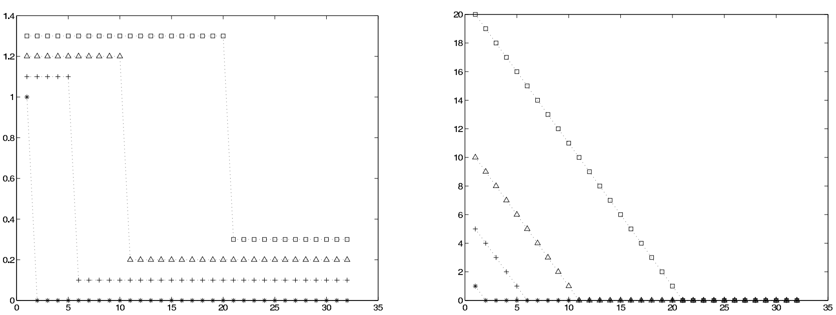

Figure 5: Discrete splines $(\boldsymbol{A}_n^m)_k^{\mathrm{T}}$, $k = 1, 5, 10, 20$, for $n = 32$ and $m = 1$ (left), $m = 2$ (right). For $m = 1$, we have added 0.1, 0.2 and 0.3 to the last columns to better visualize the discrete step functions.

Proposition 4.3 *The support vector indices* $\tilde{\Lambda}_I$ *of the solution* $\boldsymbol{U}$ *in* (19) *of the SVR problem are exactly the spline knots* Λ_I*, i.e.,*

$$\triangle^{2m} U(j - m) \neq 0 \quad \Longleftrightarrow \quad j \in \tilde{\Lambda}_I.$$

If the number of contact points $|\Lambda_I|$ is small compared to n, then $\boldsymbol{c}$ has only a small number of nonzero coefficients and (19) provides us with a *sparse representation* of $\boldsymbol{U}$. This can also be seen by noting that our SVR problem (18) means to find $\boldsymbol{U} = \boldsymbol{K}_0 \boldsymbol{c}$ such that the equality constraints are fulfilled and

$$\frac{1}{2} \|\boldsymbol{F} - \boldsymbol{U}\|^2_{\mathcal{W}^m_{2,0}} + \lambda \|\boldsymbol{c}\|_1 \quad \to \min .$$

Compare with [9] in a general SVR context. In contrast to the 2–norm, the 1–norm of $\boldsymbol{c}$ in the penalty term implies for sufficiently large λ that some of the coefficients $c(j)$ are 0. This implies a sparse representation of $\boldsymbol{U}$ from another point of view.

Finally, we see by (16) and (8) that

$$\boldsymbol{u} = \boldsymbol{D}_n^m \boldsymbol{A}_n^m (\boldsymbol{A}_n^m)^{\mathrm{T}} \boldsymbol{c} = (\boldsymbol{A}_n^m)^{\mathrm{T}} \boldsymbol{c} \tag{20}$$

is the corresponding sparse representation of our original solution $\boldsymbol{u}$. By Proposition 2.2 i) we have that $\boldsymbol{D}_{m,n}(\boldsymbol{A}_n^m)^{\mathrm{T}} = (-1)^m (\boldsymbol{I}_{n-m} | \boldsymbol{0}_{n-m,m})$ so that the first $n - m$ columns of $(\boldsymbol{A}_n^m)^{\mathrm{T}}$ are splines of degree $m - 1$ with one inner knot and the last m columns are polynomials in Π_{m-1}. For $m = 1$ and 2 some columns of $(\boldsymbol{A}_n^m)^{\mathrm{T}}$ are illustrated in Figure 5. In the context of sparse representation, the following observation is interesting: by (20), (8) and Proposition 2.2 i) and iii), our original problem (5) can be rewritten as

$$\frac{1}{2} \|\boldsymbol{f} - (\boldsymbol{A}_n^m)^{\mathrm{T}} \boldsymbol{c}\|_2^2 + \lambda \|(\boldsymbol{I}_{n-m} | \boldsymbol{0}_{n-m,m})\, \boldsymbol{c}\|_1 \quad \to \quad \min . \tag{21}$$

Remark 4.4 *Finally, let us mention that a continuous version of our considerations reads as follows: For a function* $u := \Phi_u^{(2m)}$ *we have that* $\Phi_u = k * u$*, where* k *is the causal fundamental solution of the* $2m$*–th derivative operator, i.e., the spline* $k(x) = x_+^{2m-1}$*. If* $\boldsymbol{u}$ *plays the discrete role of* u *then our discrete function* $(\boldsymbol{U}_I^T, \boldsymbol{U}_R^T)^T = \boldsymbol{A}_n^m \boldsymbol{u} = \boldsymbol{K}\boldsymbol{D}_n^m \boldsymbol{u}$ *plays the role of* $U := \Phi_u^{(m)} = k * u^{(m)}$*.*

4.3 Denoising Example

In this section, we show the performance of our approach (5) and (15) by a denoising example. We are mainly interested in the behaviour for various differentiation orders m. Our aim is to demonstrate the spline interpolation with variable knots for various m and not to create an optimal denoising method. To this end, we have used the signal shown in Figure 6 (top, left) and have added white Gaussian noise. First, we have determined the optimal parameters λ with respect to the maximal signal–to–noise–ratio (SNR) defined by $\mathrm{SNR}(g, u) := 10 \log_{10} \left(\frac{\|g\|_2^2}{\|g-u\|_2^2} \right)$ with original signal g. For the solution of the quadratic problem (15) we have applied the Matlab quadratic programming routine which is based on an active set method. Then we compared the quality of the results obtained for various m. The following table contains the results for λ, the SNR and the peak signal–to–noise–ratio (PSNR) defined by $\mathrm{PSNR}(g, u) := 10 \log_{10} \left(\frac{n\|g\|_\infty^2}{\|g-u\|_2^2} \right)$, where n denotes the number of pixels. The noisy signal in Figure 6 (top,right) has SNR 6.94 and PSNR 10.72.

m	λ	SNR	PSNR
1	20.2	16.00	19.78
2	57.8	18.41	22.18
3	275.0	17.97	21.69
4	1453.1	17.22	20.99

The corresponding signal plots are given in Figure 6. For this signal the methods with orders $m \geq 2$ perform better than the usual method with $m = 1$ where the the linear method ($m = 2$) achieves the best restoration. In general higher order methods with l_1 regularization term neglect the staircasing effect appearing in the piecewise constant approximation with $m = 1$ and preserve on the other hand local singularities better than linear methods with quadratic regularization term. Various other examples for the denoising of signals by solving (5) were presented in [26].

5 Generalization to Two Dimensions

In this section, we briefly consider a possible generalization of our concept to two dimensions. This may be considered as starting point for future

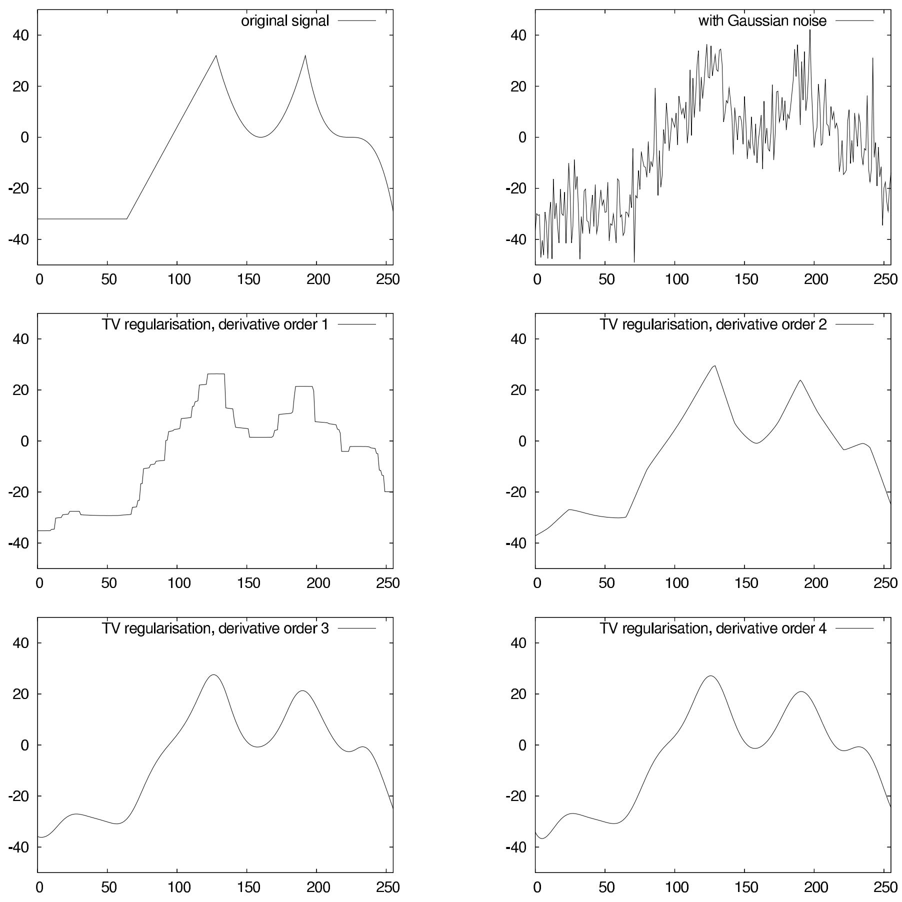

Figure 6: Denoising results with (5). Top left: original signal. Top right: noisy signal. Middle left: denoised signal for $m = 1$. Middle right: denoised signal for $m = 2$. Bottom left: denoised signal for $m = 3$. Bottom right: denoised signal for $m = 4$.

research.

Concerning *first order derivatives*, we consider the ROF model

$$\frac{1}{2}\int_\Omega (u(x)-f(x))^2 + \lambda|\nabla u|\,\mathrm{d}x \quad \to \quad \min \tag{22}$$

and the model

$$\frac{1}{2}\int_\Omega (u(x)-f(x))^2 + \lambda(|u_x|+|u_y|)\,\mathrm{d}x \quad \to \quad \min \tag{23}$$

treated, e.g., in [12]. Of course the second model is not rotationally invariant.

In the following, we restrict our attention for simplicity to quadratic $n \times n$ images and reshape them columnwise into a vector of length $N = n^2$. We discretize the first order derivatives as proposed by Chambolle in [1]. To this end, we introduce the gradient matrix

$$\boldsymbol{\mathcal{D}} := \begin{pmatrix} \boldsymbol{I}_n \otimes \boldsymbol{D}_n^0 \\ \boldsymbol{D}_n^0 \otimes \boldsymbol{I}_n \end{pmatrix} \in \mathbb{R}^{2N,N} \quad \text{with} \quad \boldsymbol{D}_n^0 := \begin{pmatrix} \boldsymbol{D}_{n,1} \\ \boldsymbol{0}_{1,n} \end{pmatrix}$$

and the Kronecker product $\otimes$. The matrix $\boldsymbol{\mathcal{D}}$ has rank $N-1$ and $\boldsymbol{\mathcal{D}}^{\mathrm{T}}$ plays the role of $-\mathrm{div} = \nabla^*$. Further, we have that $\triangle_N := \boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{\mathcal{D}}$ is the finite difference discretization of the Laplace operator with the five point scheme and Neumann boundary conditions and that

$$\begin{aligned} \mathcal{R}(\boldsymbol{\mathcal{D}}^{\mathrm{T}}) &= \mathcal{R}(\triangle_N) = \{\boldsymbol{f} \in \mathbb{R}^N : \sum_{j=1}^N f(j) = 0\}, \qquad (24) \\ \mathcal{N}(\boldsymbol{\mathcal{D}}) &= \mathcal{N}(\triangle_N) = \{\mu\,\boldsymbol{1}_{N,1} : \mu \in \mathbb{R}\} = \Pi_0. \end{aligned}$$

Finally, the discrete version of $|\nabla u| = (u_x^2 + u_y^2)^{1/2}$ reads $|\boldsymbol{\mathcal{D}}\boldsymbol{u}|$, where

$$\left|\begin{pmatrix} \boldsymbol{F}^1 \\ \boldsymbol{F}^2 \end{pmatrix}\right| := \left((\boldsymbol{F}^1)^2 + (\boldsymbol{F}^2)^2\right)^{1/2} = \left(\boldsymbol{F}^1 \circ \boldsymbol{F}^1 + \boldsymbol{F}^2 \circ \boldsymbol{F}^2\right)^{1/2} \in \mathbb{R}^N$$

and $\circ$ denotes the componentwise vector product. Now we can discretize (22) and (23) by

$$\frac{1}{2}\,\|\boldsymbol{f}-\boldsymbol{u}\|_2^2 + \lambda\|\,|\boldsymbol{\mathcal{D}}\boldsymbol{u}|\,\|_1 \tag{25}$$

and

$$\frac{1}{2}\,\|\boldsymbol{f}-\boldsymbol{u}\|_2^2 + \lambda\|\boldsymbol{\mathcal{D}}\boldsymbol{u}\|_1, \tag{26}$$

respectively. Then, by the dual approach, see, e.g. [1, 25], we obtain that $\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{V}$, where $\boldsymbol{V}$ is the solution of

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{f} - \boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{V}\|_2^2 \quad \to \quad \min \\ \text{s.t.} \quad &\|\,|\boldsymbol{V}|\,\|_\infty \le \lambda, \quad \text{in case (25)}, \qquad (27) \\ \text{s.t.} \quad &\|\boldsymbol{V}\|_\infty \le \lambda, \quad \text{in case (26)}. \qquad (28) \end{aligned}$$

The first minimization problem can be solved for example by using Chambolle's semi–implicit gradient descent algorithm [1], while the second problem can be solved by standard QP methods. An example for the solution of both problems is presented at the bottom of Figure 8. By the absence of rotation invariance, the solution of the second problem shows harder segmentation effects in x and y directions.

In the following, we assume that $\boldsymbol{f} \in \mathcal{R}(\boldsymbol{\mathcal{D}}^{\mathrm{T}})$, i.e., $\boldsymbol{f} = \boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{F}$ for some $\boldsymbol{F} \in \mathbb{R}^{2N}$. Otherwise we consider $\boldsymbol{f}-\mathrm{mean}(\boldsymbol{f})\mathbf{1}_{N,1}$. Then, since $\boldsymbol{\mathcal{D}u} = \boldsymbol{\mathcal{D}u}_{\mathcal{R}}$, and

$$\frac{1}{2}\|\boldsymbol{f}-\boldsymbol{u}\|_2^2 = \frac{1}{2}\|\boldsymbol{f}-\boldsymbol{u}_{\mathcal{R}}\|_2^2 + \frac{1}{2}\|\boldsymbol{u}_{\mathcal{N}}\|_2^2,$$

where $\boldsymbol{u}_{\mathcal{R}}$ is the orthogonal projection onto $\mathcal{R}(\boldsymbol{\mathcal{D}}^{\mathrm{T}})$ and $\boldsymbol{u}_{\mathcal{N}}$ the orthogonal projection onto $\mathcal{N}(\boldsymbol{\mathcal{D}}^{\mathrm{T}})$, it follows that the minimizer $\boldsymbol{u}$ of (25) and (26) is also in $\mathcal{R}(\boldsymbol{\mathcal{D}}^{\mathrm{T}})$. Now $\boldsymbol{U} = \boldsymbol{F} - \boldsymbol{V}$ solves the problem

$$\begin{aligned} &\frac{1}{2}\|\boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{U}\|_2^2 \quad \to \quad \min \\ \text{s.t.} \quad &\|\,|\boldsymbol{F}-\boldsymbol{U}|\,\|_\infty \le \lambda, \quad \text{in case (25)}, \\ \text{s.t.} \quad &\|\boldsymbol{F}-\boldsymbol{U}\|_\infty \le \lambda, \quad \text{in case (26)}. \end{aligned}$$

With respect to Remark 3.3 we note that the discrete G–norm defined for $\boldsymbol{v} \in \mathcal{R}(\boldsymbol{\mathcal{D}}^{\mathrm{T}})$ by $\|\boldsymbol{v}\|_G := \inf_{\boldsymbol{v}=\boldsymbol{\mathcal{D}}^{\mathrm{T}}\boldsymbol{V}} \|\,|\boldsymbol{V}|\,\|_\infty$ plays the role of the $\mathcal{W}_1(\boldsymbol{D}_{n,1})'$ norm.

For *higher order derivatives* even the choice of an appropriate disretization which preserves the basic integral identities satisfied by the continuous differential operators is a nontrivial question, see, e.g., [13]. However, operators of higher order were considered in image processing, e.g., in [5, 2, 11, 15, 20, 24, 34, 25]. Here we restrict our attention to

$$\frac{1}{2}\int_\Omega (u(x)-f(x))^2 + \lambda|\triangle u|\,\mathrm{d}x \quad \to \quad \min.$$

As discretization we choose

$$\frac{1}{2}\|\boldsymbol{f}-\boldsymbol{u}\|_2^2 + \lambda\|\triangle_D \boldsymbol{u}\|_1 \quad \to \quad \min \tag{29}$$

where $\triangle_D$ denotes the finite difference discretization of the Laplace operator with the five point scheme and Dirichlet boundary conditions. Then $\triangle_D$ is invertible. The dual approach to (29) leads with $\boldsymbol{f} = \triangle_D \boldsymbol{F}$ and $\boldsymbol{u} = \triangle_D \boldsymbol{U}$ to the contact problem

$$\begin{aligned} &\frac{1}{2}\|\triangle_D \boldsymbol{U}\|_2^2 \quad \to \quad \min \\ \text{s.t.} \quad &\|\boldsymbol{F}-\boldsymbol{U}\,\|_\infty \le \lambda, \end{aligned} \tag{30}$$

which can be solved by standard QP methods. An example for the solution of this problem in shown at the top of Figure 8. The solution contains

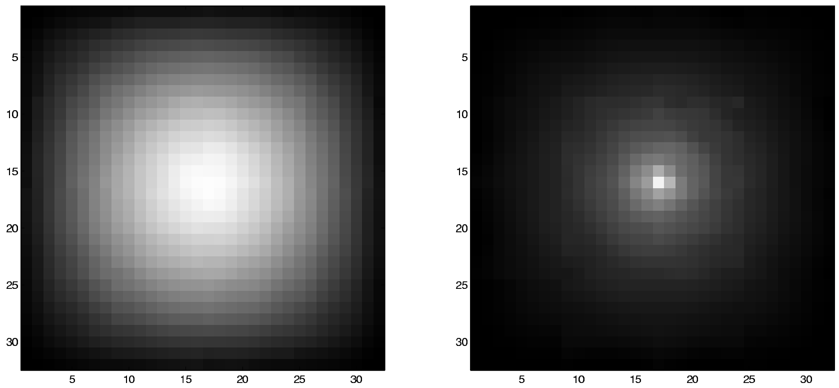

Figure 7: Column 528 of $\triangle_D^{-2}$ (left) and of $\triangle_D^{-1}$ (right) for $n = 32$.

some artefacts in form of white points which were also mentioned in [34]. Therefore the approach (29) seems to be not suited for applications in image processing. Obviously, $\triangle_D^{-2}$ is a reproducing kernel in $\mathbb{R}^N$ equipped with the norm given by the minimization term and $\boldsymbol{U} = \triangle_D^{-2}\boldsymbol{c}$ and $\boldsymbol{u} = \triangle_D^{-1}\boldsymbol{c}$ are in general sparse representations. The images corresponding to a central row of $\triangle_D^{-2}$ and $\triangle_D^{-1}$ are depicted in Figure 7.

With respect to the kernel $\triangle_D^{-2}$ let us finally note the following remark.

Remark 5.1 (Thin Plate Splines)
The so–called thin plate spline [8] $K(x) := \frac{1}{8\pi}|x|^2 \ln|x|$ *is the fundamental solution of the biharmonical operator* $\triangle^2$. *For appropriately chosen* x_j *the solution of*

$$\frac{1}{2}\sum_{j=1}^{N}(f(x_j) - u(x_j))^2 + \lambda \int_\Omega u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 \,\mathrm{d}x \quad \to \quad \min$$

has the form $u(x) = \sum_{j=1}^{N} c_j K(x - x_j) + a_0 + a_1 x + a_2 y$.

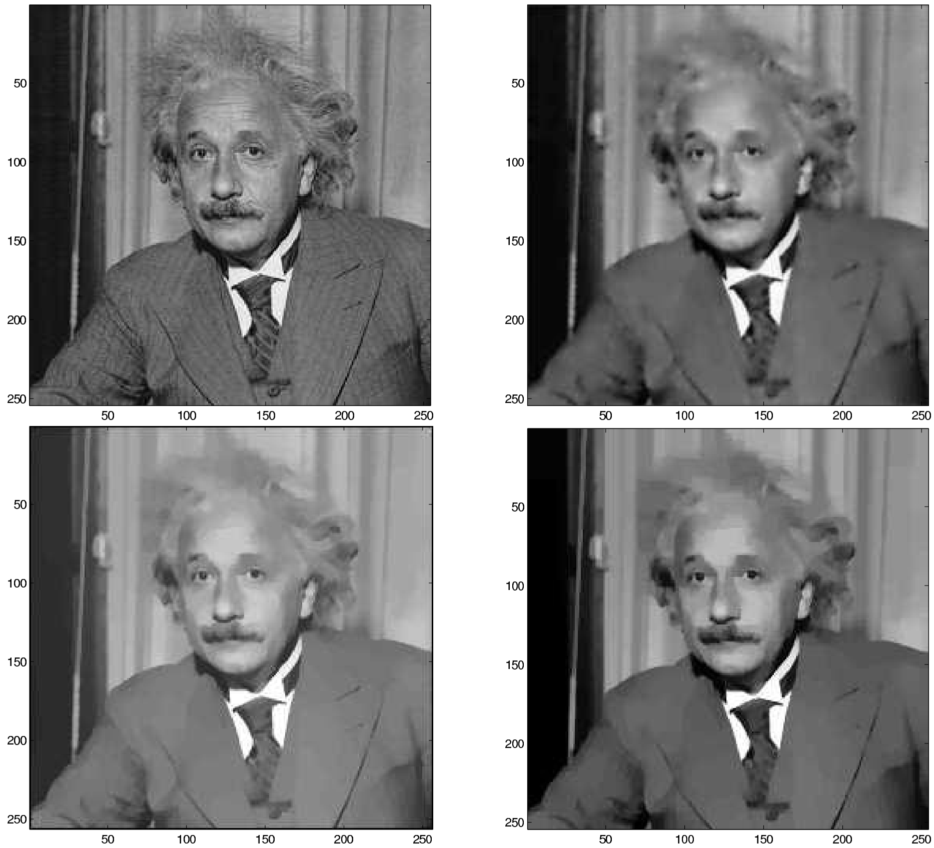

Figure 8: Top: Original 256×256 image (left). Solution of (30) (right). The image involves artefacts (white points). Bottom: Solution of (27) (left). Solution of (28) (right). The right-hand image shows a stronger segmentation in x and y direction. All problem were solved with $\lambda = 10$. For problem (27) we have used the semi–implicit gradient descent algorithm [1]. Problems (30) and (28) were computed by the ILOG CPLEX Barrier Optimizer version 7.5. This routine uses a modification of the primal–dual predictor–corrector interior point algorithm described in [19].

6 Conclusions

We have shown the equivalence of the following problems in a discrete 1D setting:

i) minimzation of a functional with quadratic data term and TV regularization term with higher order derivatives,

ii) spline interpolation with variable knots depending on the input data and the regularization parameter,

iii) hard margin SVR in the discrete counterpart of the Sobolev space $W_{2,0}^m$,

iv) sparse representation in terms of fundamental splines with penalization the of l_1 norm of the coefficients.

Based on (6) a slightly different approach which handles the boundary conditions in advance (as done in 2D) is possible. Moreover, more general spline concepts as those of exponential splines, see, e.g., [28] and other data terms incorporating only few knots or related to other than Gaussian white noise can be considered in a similar way. Finally, the 2D setting deserves stronger investigation.

References

[1] A. Chambolle. An algorithm for total variation minimization and applications. *Journal of Mathematical Imaging and Vision*, (20):89–97, 2004.

[2] A. Chambolle and P.-L. Lions. Image recovery via total variation minimization and related problems. *Numerische Mathematik*, 76:167–188, 1997.

[3] R. H. Chan, C. W. Ho, and M. Nikolova. Salt-and-pepper noise removal by median noise detectors and detail preserving regularization. *IEEE Transactions on Image Processing*, page to appear.

[4] T. F. Chan, G. H. Golub, and P. Mulet. A nonlinear primal–dual method for total-variation based image restoration. *SIAM Journal on Scientific Computing*, 20(6):1964–1977, 1999.

[5] T. F. Chan, A. Marquina, and P. Mulet. High-order total variation-based image restoration. *SIAM Journal on Scientific Computing*, 22(2):503–516, 2000.

[6] P. L. Davies and A. Kovac. Local extremes, runs, strings and multiresolution. *Annals of Statistics*, 29:1–65, 2001.

[7] S. Didas. Higher order variational methods for noise removal in signals and images. Diplomarbeit, Universität des Saarlandes, 2004.

[8] J. Duchon. Splines minimizing rotation-invariant seminorms in sobolev spaces. In *Constructive Theory of Functions of Several Variables*, pages 85–100, Berlin, 1997. Springer–Verlag.

[9] F. Girosi. An equivalence between sparse approximation and support vector machines. *Neural computation*, 10(6):1455–1480, 1998.

[10] W. Hinterberger, M. Hintermüller, K. Kunisch, M. von Oehsen, and O. Scherzer. Tube methods for BV regularization. *Journal of Mathematical Imaging and Vision*, 19:223 – 238, 2003.

[11] W. Hinterberger and O. Scherzer. Variational methods on the space of functions of bounded Hessian for convexification and denoising. Technical report, University of Innsbruck, Austria, 2003.

[12] W. Hintermüller and K. Kunisch. Total bounded variation regularization as a bilaterally constrained optimization problem. *SIAM Journal on Applied Mathematics*, 64(4):1311–1333, May 2004.

[13] J. M. Hyman and M. J. Shashkov. Natural discretizations for the divergence, gradient, and curl on logically rectangular grids. *Comput. Math. Appl.*, 33(4):81–104, 1997.

[14] G. S. Kimmeldorf and G. Wahba. Some results on Tchebycheffian spline functions. *J. Anal. Appl.*, 33:82–95, 1971.

[15] M. Lysaker, A. Lundervold, and X. Tai. Noise removal using fourth-order partial differential equations with applications to medical magnetic resonance images in space and time. *IEEE Transactions on Image Processing*, 12(12):1579 – 1590, 2003.

[16] E. Mammen and S. van de Geer. Locally adaptive regression splines. *Annals of Statistics*, 25(1):387–413, 1997.

[17] O. L. Mangasarian and L. L. Schumaker. Discrete splines via mathematical programming. *SIAM Journal on Control*, 9(2):174–183, 1971.

[18] O. L. Mangasarian and L. L. Schumaker. Best summation formulae and discrete splines via mathematical programming. *SIAM Journal on Numerical Analysis*, 10(3):448–459, 1973.

[19] S. Mehrotra. On the implementation of a primal-dual interior point method. *SIAM Journal on Optimization*, 2(4):575–601, 1992.

[20] M. Nielsen, L. Florack, and R. Deriche. Regularization, scale-space and edge detection filters. *Journal of Mathematical Imaging and Vision*, 7:291–307, 1997.

[21] M. Nikolova. A variational approach to remove outliers and impulse noise. *Journal of Mathematical Imaging and Vision*, 20:99–120, 2004.

[22] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, Princeton, 1970.

[23] L. I. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.

[24] C. Schnörr. A study of a convex variational diffusion approach for image segmentation and feature extraction. *Journal of Mathematical Imaging and Vision*, 8(3):271–292, 1998.

[25] G. Steidl. A note on the dual treatment of higher order regularization functionals. *Computing*, 2005, to appear.

[26] G. Steidl, S. Didas, and J. Neumann. Relations between higher order TV regularization and support vector regression. In R. Kimmel, N. Sochen, and J. Weickert, editors, *Scale-Space and PDE Methods in Computer Vision*, volume 3459 of *Lecture Notes in Computer Science*, pages 515–527. Springer, Berlin, 2005.

[27] G. Steidl, J. Weickert, T. Brox, P. Mrázek, and M. Welk. On the equivalence of soft wavelet shrinkage, total variation diffusion, total variation regularization, and SIDEs. *SIAM Journal on Numerical Analysis*, 42(2):686–713, 2004.

[28] M. Unser and T. Blu. Generalized smoothing splines and the optimal discretization of the Wiener filter. *IEEE Transactions on Signal Processing*, 53(6):2146–2159, 2005.

[29] V. N. Vapnik. *Statistical Learning Theory.* John Wiley and Sons, Inc., 1998.

[30] C. R. Vogel. *Computational Methods for Inverse Problems.* SIAM, Philadelphia, 2002.

[31] G. Wahba. *Spline Models for Observational Data.* SIAM, Philadelphia, 1990.

[32] M. Welk, J. Weickert, and G. Steidl. A four-pixel scheme for singular differential equations. In R. Kimmel, N. Sochen, and J. Weickert, editors, *Scale-Space and PDE Methods in Computer Vision*, Lecture Notes in Computer Science. Springer, Berlin, 2005, to appear.

[33] A. M. Yip and F. Park. Solution dynamics, causality, and critical behaviour of the regularization parameter in total variation denoising problems. UCLA Report, 2003.

[34] Y.-L. You and M. Kaveh. Fourth-order partial differential equations for noise removal. *IEEE Transactions on Image Processing*, 9(10):1723–1730, 2000.

Preprint Series DFG-SPP 1114

http://www.math.uni-bremen.de/zetem/DFG-Schwerpunkt/preprints/

Reports

[1] Werner Horbelt, Jens Timmer, and Henning U. Voss. Parameter estimation in nonlinear delayed feedback systems from noisy data. 2002 May. ISBN 3-88722-530-9.

[2] Andreas Martin. Propagation of singularities. 2002 July. ISBN 3-88722-533-3.

[3] Thorsten G. Müller and Jens Timmer. Fitting parameters in partial differential equations from partially observed noisy data. 2002 August. ISBN 3-88722-536-8.

[4] Gabriele Steidl, Stephan Dahlke, and Gerd Teschke. Coorbit spaces and banach frames on homogeneous spaces with applications to the sphere. 2002 August. ISBN 3-88722-537-6.

[5] Jens Timmer, Thorsten G. Müller, I. Swameye, O. Sandra, and U. Klingmüller. Modeling the non-linear dynamics of cellular signal transduction. 2002 September. ISBN 3-88722-539-2.

[6] M. Thiel, M.C. Romano, U. Schwarz, Jürgen Kurths, and Jens Timmer. Surrogate based hypothesis test without surrogates. 2002 September. ISBN 3-88722-540-6.

[7] Karsten Keller and H. Lauffer. Symbolic analysis of high-dimensional time series. 2002 September. ISBN 3-88722-538-4.

[8] F. Friedrich, Gerhard Winkler, O. Wittich, and V. Liebscher. Elementary rigorous introduction to exact sampling. 2002 October. ISBN 3-88722-541-4.

[9] S. Albeverio and D. Belomestny. Reconstructing the intensity of non-stationary poisson. 2002 November. ISBN 3-88722-544-9.

[10] O. Treiber, F. Wanninger, Hartmut Führ, W. Panzer, Gerhard Winkler, and D. Regulla. An adaptive algorithm for the detection of microcalcifications in simulated low-dose mammography. 2002 November. ISBN 3-88722-545-7.

[11] M. Peifer, Jens Timmer, and Henning U. Voss. Nonparametric identification of nonlinear oscillating systems. 2002 November. ISBN 3-88722-546-5.

[12] Sergej M. Prigarin and Gerhard Winkler. Numerical solution of boundary value problems for stochastic differential equations on the basis of the gibbs sampler. 2002 November. ISBN 3-88722-549-X.

[13] Andreas Martin, Sergej M. Prigarin, and Gerhard Winkler. Exact numerical algorithms for linear stochastic wave equation and stochastic klein-gordon equation. 2002 November. ISBN 3-88722-547-3.

[14] Andreas Groth. Estimation of periodicity in time series by ordinal analysis with application to speech. 2002 November. ISBN 3-88722-550-3.

[15] Henning U. Voss, Jens Timmer, and Jürgen Kurths. Nonlinear dynamical system identification from uncertain and indirect measurements. 2002 December. ISBN 3-88722-548-1.

[16] Ulrich Clarenz, Marc Droske, and Martin Rumpf. Towards fast non-rigid registration. 2002 December. ISBN 3-88722-551-1.

[17] Ulrich Clarenz, Stefan Henn, Martin Rumpf, and Kristian Witsch. Relations between optimization and gradient flow with applications to image registration. 2002 December. ISBN 3-88722-552-X.

[18] Marc Droske and Martin Rumpf. A variational approach to non-rigid morphological registration. 2002 December. ISBN 3-88722-553-8.

[19] Tobias Preußer and Martin Rumpf. Extracting motion velocities from 3d image sequences and spatio-temporal smoothing. 2002 December. ISBN 3-88722-555-4.

[20] K. Mikula, Tobias Preußer, and Martin Rumpf. Morphological image sequence processing. 2002 December. ISBN 3-88722-556-2.

[21] Volker Reitmann. Observation stability for controlled evolutionary variational inequalities. 2003 January. ISBN 3-88722-557-0.

[22] Karsten Koch. A new family of interpolating scaling vectors. 2003 January. ISBN 3-88722-558-9.

[23] Andreas Martin. Small ball asymptotics for the stochastic wave equation. 2003 January. ISBN 3-88722-559-7.

[24] Peter Maaß, Torsten Köhler, Rosa Costa, U. Parlitz, Jan Kalden, U. Wichard, and C. Merkwirth. Mathematical methods for forecasting bank transaction data. 2003 January. ISBN 3-88722-569-4.

[25] D. Belomestny and H. Siegel. Stochastic and self-similar nature of highway traffic data. 2003 February. ISBN 3-88722-568-6.

[26] Gabriele Steidl, Joachim Weickert, T. Brox, Pavel Mrázek, and M. Welk. On the equivalence of soft wavelet shrinkage, total variation diffusion, and sides. 2003 February. ISBN 3-88722-561-9.

[27] J. Polzehl and V. Spokoiny. Local likelihood modeling by adaptive weights smoothing. 2003 February. ISBN 3-88722-564-3.

[28] Ingo Stuke, Til Aach, Cicero Mota, and Erhardt Barth. Estimation of multiple motions: regularization and performance evaluation. 2003 February. ISBN 3-88722-565-1.

[29] Ingo Stuke, Til Aach, Cicero Mota, and Erhardt Barth. Linear and regularized solutions for multiple motions. 2003 February. ISBN 3-88722-566-X.

[30] Werner Horbelt and Jens Timmer. Asymptotic scaling laws for precision of parameter estimates in dynamical systems. 2003 February. ISBN 3-88722-567-8.

[31] Rainer Dahlhaus and Suhasini Subba Rao. Statistical inference of time-varying arch processes. 2003 April. ISBN 3-88722-572-4.

[32] Gerhard Winkler, A. Kempe, V. Liebscher, and O. Wittich. Parsimonious segmentation of time series by potts models. 2003 April. ISBN 3-88722-573-2.

[33] Ronny Ramlau and Gerd Teschke. Regularization of sobolev embedding operators and applications. 2003 April. ISBN 3-88722-574-0.

[34] Kristian Bredies, Dirk A. Lorenz, and Peter Maaß. Mathematical concepts of multiscale smoothing. 2003 April. ISBN 3-88722-575-9.

[35] Andreas Martin, Sergej M. Prigarin, and Gerhard Winkler. Exact and fast numerical algorithms for the stochastic wave equation. 2003 May. ISBN 3-88722-576-7.

[36] D. Maraun, Werner Horbelt, H. Rust, Jens Timmer, H.P. Happersberger, and F. Drepper. Identification of rate constants and non-observable absorption spectra in nonlinear biochemical reaction dynamics. 2003 May. ISBN 3-88722-577-5.

[37] Quanbo Xie, Matthias Holschneider, and Michail Kulesh. Some remarks on linear diffeomorphisms in wavelet space. 2003 July. ISBN 3-88722-582-1.

[38] Mamadou Sanou Diallo, Matthias Holschneider, Michail Kulesh, Frank Scherbaum, and Frank Adler. Characterization of seismic waves polarization attributes using continuous wavelet transforms. 2003 July. ISBN 3-88722-581-3.

[39] Thomas Maiwald, Matthias Winterhalder, A. Aschenbrenner-Scheibe, Henning U. Voss, A. Schulze-Bonhage, and Jens Timmer. Comparison of three nonlinear seizure prediction methods by means of the seizure prediction characteristic. 2003 September. ISBN 3-88722-594-5.

[40] Michail Kulesh, Matthias Holschneider, Mamadou Sanou Diallo, Quanbo Xie, and Frank Scherbaum. Modeling of wave dispersion using continuous wavelet transforms. 2003 October. ISBN 3-88722-595-3.

[41] A.G.Rossberg, K.Bartholomé, and J.Timmer. Data-driven optimal filtering for phase and frequency of noisy oscillations: Application to vortex flow metering. 2004 January. ISBN 3-88722-600-3.

[42] Karsten Koch. Interpolating scaling vectors. 2004 February. ISBN 3-88722-601-1.

[43] Olaf Hansen, Silva Fischer, and Ronny Ramlau. Regularization of mellin-type inverse problems with an application to oil engeneering. 2004 February. ISBN 3-88722-602-X.

[44] Til Aach, Ingo Stuke, Cicero Mota, and Erhardt Barth. Estimation of multiple local orientations in image signals. 2004 February. ISBN 3-88722-607-0.

[45] Cicero Mota, Til Aach, Ingo Stuke, and Erhardt Barth. Estimation of multiple orientations in multi-dimensional signals. 2004 February. ISBN 3-88722-608-9.

[46] Ingo Stuke, Til Aach, Erhardt Barth, and Cicero Mota. Analysing superimposed oriented patterns. 2004 February. ISBN 3-88722-609-7.

[47] Henning Thielemann. Bounds for smoothness of refinable functions. 2004 February. ISBN 3-88722-610-0.

[48] Dirk A. Lorenz. Variational denoising in besov spaces and interpolation of hard and soft wavelet shrinkage. 2004 March. ISBN 3-88722-605-4.

[49] Volker Reitmann and Holger Kantz. Frequency domain conditions for the existence of almost periodic solutions in evolutionary variational inequalities. 2004 March. ISBN 3-88722-606-2.

[50] Karsten Koch. Interpolating scaling vectors: Application to signal and image denoising. 2004 May. ISBN 3-88722-614-3.

[51] Pavel Mrázek, Joachim Weickert, and Andrés Bruhn. On robust estimation and smoothing with spatial and tonal kernels. 2004 June. ISBN 3-88722-615-1.

[52] Dirk A. Lorenz. Solving variational problems in image processing via projections - a common view on tv denoising and wavelet shrinkage. 2004 June. ISBN 3-88722-616-X.

[53] A.G. Rossberg, K. Bartholomé, Henning U. Voss, and Jens Timmer. Phase synchronization from noisy univariate signals. 2004 August. ISBN 3-88722-617-8.

[54] Markus Fenn and Gabriele Steidl. Robust local approximation of scattered data. 2004 October. ISBN 3-88722-622-4.

[55] Henning Thielemann. Audio processing using haskell. 2004 October. ISBN 3-88722-623-2.

[56] Matthias Holschneider, Mamadou Sanou Diallo, Michail Kulesh, Frank Scherbaum, Matthias Ohrnberger, and Erika Lück. Characterization of dispersive surface wave using continuous wavelet transforms. 2004 October. ISBN 3-88722-624-0.

[57] Mamadou Sanou Diallo, Michail Kulesh, Matthias Holschneider, and Frank Scherbaum. Instantaneous polarization attributes in the time-frequency domain and wave field separation. 2004 October. ISBN 3-88722-625-9.

[58] Stephan Dahlke, Erich Novak, and Winfried Sickel. Optimal approximation of elliptic problems by linear and nonlinear mappings. 2004 October. ISBN 3-88722-627-5.

[59] Hanno Scharr. Towards a multi-camera generalization of brightness constancy. 2004 November. ISBN 3-88722-628-3.

[60] Hanno Scharr. Optimal filters for extended optical flow. 2004 November. ISBN 3-88722-629-1.

[61] Volker Reitmann and Holger Kantz. Stability investigation of volterra integral equations by realization theory and frequency-domain methods. 2004 November. ISBN 3-88722-636-4.

[62] Cicero Mota, Michael Door, Ingo Stuke, and Erhardt Barth. Categorization of transparent-motion patterns using the projective plane. 2004 November. ISBN 3-88722-637-2.

[63] Ingo Stuke, Til Aach, Erhardt Barth, and Cicero Mota. Multiple-motion estimation by block-matching using markov random fields. 2004 November. ISBN 3-88722-635-6.

[64] Cicero Mota, Ingo Stuke, Til Aach, and Erhardt Barth. Spatial and spectral analysis of occluded motions. 2004 November. ISBN 3-88722-638-0.

[65] Cicero Mota, Ingo Stuke, Til Aach, and Erhardt Barth. Estimation of multiple orientations at corners and junctions. 2004 November. ISBN 3-88722-639-9.

[66] A. Benabdallah, A. Löser, and G. Radons. From hidden diffusion processes to hidden markov models. 2004 December. ISBN 3-88722-641-0.

[67] Andreas Groth. Visualization and detection of coupling in time series by order recurrence plots. 2004 December. ISBN 3-88722-642-9.

[68] Matthias Winterhalder, Björn Schelter, Jürgen Kurths, and Jens Timmer. Sensitivity and specificity of coherence and phase synchronization analysis. 2005 January. ISBN 3-88722-648-8.

[69] Matthias Winterhalder, Björn Schelter, Wolfram Hesse, K. Schwab, Lutz Leistritz, D. Klan, R. Bauer, Jens Timmer, and H. Witte. Comparison of time series analysis techniques to detect direct and time-varying interrelations in multivariate, neural systems. 2005 January. ISBN 3-88722-643-7.

[70] Björn Schelter, Matthias Winterhalder, K. Schwab, Lutz Leistritz, Wolfram Hesse, R. Bauer, H. Witte, and Jens Timmer. Quantification of directed signal transfer within neural networks by partial directed coherence: A novel approach to infer causal time-depending influences in noisy, multivariate time series. 2005 January. ISBN 3-88722-644-5.

[71] Björn Schelter, Matthias Winterhalder, B. Hellwig, B. Guschlbauer, C.H. Lücking, and Jens Timmer. Direct or indirect? graphical models for neural oscillators. 2005 January. ISBN 3-88722-645-3.

[72] Björn Schelter, Matthias Winterhalder, Thomas Maiwald, A. Brandt, A. Schad, A. Schulze-Bonhage, and Jens Timmer. Testing statistical significance of multivariate epileptic seizure prediction methods. 2005 January. ISBN 3-88722-646-1.

[73] Björn Schelter, Matthias Winterhalder, M. Eichler, M. Peifer, B. Hellwig, B. Guschlbauer, C.H. Lücking, Rainer Dahlhaus, and Jens Timmer. Testing for directed influences in neuroscience using partial directed coherence. 2005 January. ISBN 3-88722-647-X.

[74] Dirk Lorenz and Torsten Köhler. A comparison of denoising methods for one dimensional time series. 2005 January. ISBN 3-88722-649-6.

[75] Esther Klann, Peter Maaß, and Ronny Ramlau. Tikhonov regularization with wavelet shrinkage for linear inverse problems. 2005 January.

[76] Eduardo Valenzuela-Domínguez and Jürgen Franke. A bernstein inequality for strongly mixing spatial random processes. 2005 January. ISBN 3-88722-650-X.

[77] Joachim Weickert, Gabriele Steidl, Pavel Mrázek, M. Welk, and T. Brox. Diffusion filters and wavelets: What can they learn from each other? 2005 January.

[78] M. Peifer, Björn Schelter, Matthias Winterhalder, and Jens Timmer. Mixing properties of the rössler system and consequences for coherence and synchronization analysis. 2005 January. ISBN 3-88722-651-8.

[79] Ulrich Clarenz, Marc Droske, Stefan Henn, Martin Rumpf, and Kristian Witsch. Computational methods for nonlinear image registration. 2005 January.

[80] Ulrich Clarenz, Nathan Litke, and Martin Rumpf. Axioms and variational problems in surface parameterization. 2005 January.

[81] Robert Strzodka, Marc Droske, and Martin Rumpf. Image registration by a regularized gradient flow - a streaming implementation in dx9 graphics hardware. 2005 January.

[82] Marc Droske and Martin Rumpf. A level set formulation for willmore flow. 2005 January.

[83] Hanno Scharr, Ingo Stuke, Cicero Mota, and Erhardt Barth. Estimation of transparent motions with physical models for additional brightness variation. 2005 February.

[84] Kai Krajsek and Rudolf Mester. Wiener-optimized discrete filters for differential motion estimation. 2005 February.

[85] Ronny Ramlau and Gerd Teschke. Tikhonov replacement functionals for iteratively solving nonlinear operator equations. 2005 March.

[86] Matthias Mühlich and Rudolf Mester. Derivation of the tls error matrix covariance for orientation estimation using regularized differential operators. 2005 March.

[87] Mamadou Sanou Diallo, Michail Kulesh, Matthias Holschneider, Kristina Kurennaya, and Frank Scherbaum. Instantaneous polarization attributes based on adaptive covariance method. 2005 March.

[88] Robert Strzodka and Christoph S. Garbe. Real-time motion estimation and visualization on graphics cards. 2005 April.

[89] Matthias Holschneider and Gerd Teschke. On the existence of optimally localized wavelets. 2005 April.

[90] Gerd Teschke. Multi-frame representations in linear inverse problems with mixed multi-constraints. 2005 April.

[91] Rainer Dahlhaus and Suhasini Subba Rao. A recursive online algorithm for the estimation of time-varying arch parameters. 2005 April.

[92] Suhasini Subba Rao. On some nonstationary, nonlinear random processes and their stationary approximations. 2005 April.

[93] Suhasini Subba Rao. Statistical analysis of a spatio-temporal model with location dependent parameters and a test for spatial stationarity. 2005 April.

[94] Piotr Fryzlewicz, Theofanis Sapatinas, and Suhasini Subba Rao. Normalised least squares estimation in locally stationary arch models. 2005 April.

[95] Piotr Fryzlewicz, Theofanis Sapatinas, and Suhasini Subba Rao. Haar-fisz technique for locally stationary volatility estimation. 2005 April.

[96] Suhasini Subba Rao. On multiple regression models with nonstationary correlated errors. 2005 April.

[97] Sébastien Van Bellegam and Rainer Dahlhaus. Semiparametric estimation by model selection for locally stationary processes. 2005 April.

[98] M. Griebel, Tobias Preußer, Martin Rumpf, A. Schweitzer, and A. Telea. Flow field clustering via algebraic multigrid. 2005 April.

[99] Marc Droske and Wolfgang Ring. A mumford-shah level-set approach for geometric image registration. 2005 April.

[100] M. Diehl, R. Küsters, and Hanno Scharr. Simultaneous estimation of local and global parameters in image sequences. 2005 April.

[101] Hanno Scharr, M.J. Black, and H.W. Haussecker. Image statistics and anisotropic diffusion. 2005 April.

[102] Hanno Scharr, M. Felsberg, and P.E. Forssén. Noise adaptive channel smoothing of low-dose images. 2005 April.

[103] Hanno Scharr and R. Küsters. A linear model for simultaneous estimation of 3d motion and depth. 2005 April.

[104] Hanno Scharr and R. Küsters. Simultaneous estimation of motion and disparity: Comparison of 2-, 3- and 5-camera setups. 2005 April.

[105] Christoph Bandt. Ordinal time series analysis. 2005 April.

[106] Christoph Bandt and Faten Shiha. Order patterns in time series. 2005 April.

[107] Christoph Bandt and Bernd Pompe. Permutation entropy: a natural complexity measure for time series. 2005 April.

[108] Christoph Bandt, Gerhard Keller, and Bernd Pompe. Entropy of interval maps via permutations. 2005 April.

[109] Matthias Mühlich. Derivation of optimal equilibration transformations for general covariance tensors of random matrices. 2005 April.

[110] Rudolf Mester. A new view at differential and tensor-based motion estimation schemes. 2005 April.

[111] Kai Krajsek. Steerable filters in motion estimation. 2005 April.

[112] Matthias Mühlich and Rudolf Mester. A statistical extension of normalized convolution and its usage for image interpolation and filtering. 2005 April.

[113] Matthias Mühlich and Rudolf Mester. Unbiased errors-in-variables estimation using generalized eigensystem analysis. 2005 April.

[114] Rudolf Mester. The generalization, optimization and information–theoretic justification of filter-based and autocovariance based motion estimation. 2005 April.

[115] Rudolf Mester. On the mathematical structure of direction and motion estimation. 2005 April.

[116] Kai Krajsek and Rudolf Mester. Signal and noise adapted filters for differential motion estimation. 2005 April.

[117] Matthias Mühlich and Rudolf Mester. Optimal homogeneous vector estimation. 2005 April.

[118] Matthias Mühlich and Rudolf Mester. A fast algorithm for statistically optimized orientation estimation. 2005 April.

[119] Christoph S. Garbe, Hagen Spies, and Bernd Jähne. Estimation of surface flow and net heat flux from infrared image sequences. 2005 April.

[120] Christoph S. Garbe, Hagen Spies, and Bernd Jähne. Mixed ols-tls for the estimation of dynamic processes with a linear source term. 2005 April.

[121] Christoph S. Garbe, Hagen Spies, and Bernd Jähne. Estimation of complex motion from thermographic image sequences. 2005 April.

[122] Christoph S. Garbe, U. Schimpf, and Bernd Jähne. A surface renewal model to analyze infrared image sequences of the ocean surface for the study of air-sea heat and gas exchange. 2005 April.

[123] Bernd Jähne and Christoph S. Garbe. Towards objective performance analysis for estimation of complex motion: Analytic motion modeling, filter optimization, and test sequences. 2005 April.

[124] Hagen Spies and Christoph S. Garbe. Dense parameter fields from total least squares. 2005 April.

[125] Hagen Spies, T. Dierig, and Christoph S. Garbe. Local models for dynamic processes in image sequences. 2005 April.

[126] Hanno Scharr. Optimal derivative filter families for transparent motion estimation. 2005 April.

[127] Matthias Mühlich. Subspace estimation with uncertain and correlated data. 2005 April.

[128] Karsten Keller and Katharina Wittfeld. Distances of time series components by means of symbolic dynamics. 2005 April.

[129] Hermann Haase, Susanna Braun, Birgit Arheilger, and Michael Jünger. Symbolic wavelet analysis of cutaneaus blood flow and application in dermatology. 2005 May.

[130] Norbert Marwan, Andreas Groth, and Jürgen Kurths. Quantification of order patterns recurrence plots of event related potentials. 2005 June.

[131] Gerd Teschke. Multi–frames in thresholding iterations for nonlinear operator equations with mixed sparsity constraints. 2005 July.

[132] Pavel Mrázek and Joachim Weickert. From two-dimensional nonlinear diffusion to coupled haar wavelet shrinkage. 2005 August.

[133] Karsten Koch. Nonseparable orthonormal interpolating scaling vectors. 2005 September.

[134] Stephan Dahlke, Massimo Fornasier, Holger Rauhut, Gabriele Steidl, and Gerd Teschke. Generalized coorbit theory, banach frames, and the relation to α-modulation spaces. 2005 September.

[135] Kristian Bredies, Dirk A. Lorenz, and Peter Maaß. Equivalence of a generalized conditional gradient method and the method of surrogate functionals. 2005 September.

[136] Stephan Didas, Pavel Mrázek, and Joachim Weickert. Energy-based image simplification with nonlocal data and smoothness terms. 2005 November.

[137] Michail Kulesh, Mamadou Sanou Diallo, Matthias Holschneider, Kristina Kurennaya, Frank Krüger, Matthias Ohrnberger, and Frank Scherbaum. Polarization analysis in wavelet domain based on the adaptive covariance method. 2005 December.

[138] Stephan Dahlke, Massimo Fornasier, Thorsten Raasch, Rob Stevenson, and Manuel Werner. Adaptive frame methods for elliptic operator equations: The steepest descent approach. 2006 February.

[139] Kristina Kurennaya, Michail Kulesh, and Matthias Holschneider. Adaptive metrics in the nearest neighbours method. 2006 April.

[140] Henning Thielemann. Optimally matched wavelets. 2006 April.

[141] Stephan Dahlke, Dirk Lorenz, Peter Maass, Chen Sagiv, and Gerd Teschke. The canonical coherent states associated with quotients of the affine weyl-heisenberg group. 2006 June.

[142] Gabriele Steidl, Stephan Didas, and Julia Neumann. Splines in higher order TV regularization. 2006 June.