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The Morphological Equivalents of Relativistic and Alpha-Scale-Spaces

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Abstract. The relations between linear system theory and mathematical morphology are mainly understood on a pure convolution / dilation level. A formal connection on the level of differential or pseudo-differential equations is still missing. In our paper we close this gap. We establish the sought relation by means of infinitesimal generators, exploring essential properties of the slope and a modified Cramér transform. As an application of our general theory, we derive the morphological counterparts of relativistic scale-spaces and of α-scale-spaces for $\alpha \in [\frac{1}{2}, \infty)$. Our findings are illustrated by experiments.

Keywords: mathematical morphology, alpha-scale-spaces, relativistic scale-spaces, Cramér transform, slope transform

1 Introduction

Linear system theory and mathematical morphology are two successful and widely-used concepts in signal and image processing. It is well-known that any shift-invariant linear system can be described as a convolution that can be elegantly computed as multiplication in the Fourier domain [14]. On the other hand, morphological systems are based on dilations with a concave structuring function, which comes down to additions in the slope domain [15, 7]. First insights into the quasi-logarithmic connection between both worlds have been obtained by Burgeth and Weickert [5]: While linear system theory uses the classical algebra $(\mathbb{R}, \cdot, +)$, they showed that mathematical morphology is a system theory in the max-plus algebra $(\mathbb{R} \cup \{-\infty\}, +, \max)$. Moreover, they described this relation by means of the Cramér transform. So far, this formal connection is restricted to the level of convolutions on the linear system theory side and dilations on the morphological side.

Continuous-scale interpretations of both frameworks allow to describe linear and morphological systems in terms of partial differential equations (PDEs) or pseudo-differential equations. For example, Gaussian convolution comes down to a homogeneous diffusion equation, whose evolution in time creates the so-called Gaussian scale-space [11, 21, 12]. More recently, scale-spaces based on pseudo-differential operators have attracted attention, such as the Poisson scale-space [9], its embedding into the family of α-scale-spaces [8], and relativistic scale-spaces [4]. On the morphological side, continuous-scale versions of dilations are given by hyperbolic PDEs [2, 19].

An interesting equivalence between Gaussian scale-space and morphological dilation with a quadratic structuring function has been discovered by van den Boomgaard [17]: While Gaussians are the only separable and rotationally invariant convolution kernels, quadratic functions are the only separable and rotationally invariant structuring functions. Other formal equivalences between the (pseudo-)differential operators governing linear shift-invariant scale-spaces and morphological scale-spaces are not known so far.

The goal of our paper is to address this problem. We establish a general theory that allows to transform a scale-scale evolution from one of these worlds to the other world. This framework extends the results of Burgeth and Weickert [5] to differential and pseudo-differential operators. In particular, it enables us to derive the morphological counterparts of α-scale-spaces for $\alpha \in [\frac{1}{2}, \infty)$, and of relativistic scale-spaces.

Organisation of the Paper. Sections 2 and 3 review relevant concepts for linear shift-invariant scale-spaces and morphological scale-spaces, respectively. These facts allow us to derive our general framework in Section 4. The fifth Section applies our theory to α-scale-spaces and relativistic scale-spaces. Experiments in Section 6 illustrate the behaviour of their morphological counterparts. Our paper is concluded with a summary in Section 7.

2 Convolution Scale-Spaces

Let us consider some bounded greyscale image $f : \mathbb{R}^2 \to \mathbb{R}$. A scale-space representation of f embeds this image into a family $u(.,t)$ of gradually smoother versions, where the scale parameter ("time") t determines the amount of smoothing or image simplification: $t = 0$ yields $u(.,0) = f$, and larger values for t correspond to simpler versions of f with less structure. Reasonable scale-spaces have to satisfy a number of architectural properties, simplification qualities, and invariances [1]. Typically their evolutions w.r.t. the scale-parameter t can be expressed in terms of differential or pseudo-differential equations. In our paper we focus on scale-space evolutions that are both linear and shift-invariant. More specifically, we consider the following processes:

– **Gaussian Scale-Space.** It computes smoothed versions $u(\boldsymbol{x}, t)$ of $f(\boldsymbol{x})$ as solutions of the initial value problem

$$\partial_t u = \Delta u \qquad \text{on } \mathbb{R}^2 \times (0, \infty), \tag{1}$$

$$u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) \qquad \text{on } \mathbb{R}^2, \tag{2}$$

where Δ denotes the spatial Laplace operator. It goes back to Iijima [11, 20] and became popular in the western world by the work of Witkin [21], Koenderink [12], Lindeberg [13], and Florack [10] and many others.

– **α-Scale-Spaces.** They replace the homogeneous diffusion equation (1) by the pseudo-differential equation

$$\partial_t u = -(-\Delta)^\alpha u \tag{3}$$

with some parameter $\alpha \in (0, \infty)$. While these processes can already be found implicitly in Iijima's work [11], they became popular as scale-spaces due to a paper by Duits et al. [8]. Gaussian scale-space is recovered for $\alpha = 1$, while $\alpha = \frac{1}{2}$ gives the so-called Poisson scale-space [9]. If one renounces a maximum–minimum principle, one can also study scale-spaces for $\alpha > 1$, comprising e.g. the biharmonic scale-space for $\alpha = 2$ [6].

– **Relativistic Scale-Spaces.** Burgeth et al. [4] have advocated a generalisation of the Poisson scale-space by considering the evolution equation

$$\partial_t u = -\left(\sqrt{-\Delta + m^2} - m\right) u, \tag{4}$$

with $m \geq 0$.

Since all these processes are linear and shift-invariant, they can be expressed by convolutions with a suitable kernel k_t:

$$u(., t) = k_t * f. \tag{5}$$

Thus, we can call such a linear, shift-invariant scale-space also a *convolution scale-space*. For some of the beforementioned convolution scale-spaces, however, the kernel does not have a closed form representation in the spatial domain. One exception is Gaussian scale-space, for which the kernel is given by the Gaussian

$$g_t(\boldsymbol{x}) = \frac{1}{4\pi t} \exp\left(-\frac{|\boldsymbol{x}|^2}{4t}\right). \tag{6}$$

For kernels that do not have a closed form representation in the spatial domain, it can be convenient to use a closed form description in the Fourier domain: We define the Fourier transform by

$$\hat{u}(\boldsymbol{\nu}) := \mathcal{F}[u](\boldsymbol{\nu}) := \int_{\mathbb{R}^2} u(\boldsymbol{x})\, e^{-2\pi i \langle \boldsymbol{\nu}, \boldsymbol{x} \rangle}\, d\boldsymbol{x} \tag{7}$$

Table 1. Linear shift-invariant scale-spaces and the Fourier transform of their convolution kernels [8, 4, 3].

scale-space	Fourier transform of convolution kernel
Gaussian scale-space	$\hat{g}_t(\boldsymbol{\nu}) = \exp\left(-t\,\lvert 2\pi\boldsymbol{\nu}\rvert^2\right)$
α-scale-spaces	$\hat{a}_{\alpha,t}(\boldsymbol{\nu}) = \exp\left(-t\,\lvert 2\pi\boldsymbol{\nu}\rvert^{2\alpha}\right)$
relativistic scale-spaces	$\hat{r}_{m,t}(\boldsymbol{\nu}) = \exp\left(-\,t\big(\sqrt{\lvert 2\pi\boldsymbol{\nu}\rvert^2 + m^2} - m\big)\right)$

where $i^2 = -1$ and $\langle .,. \rangle$ denotes the Euclidean inner product. Then the Fourier transforms of the convolution kernels of the individual scale-spaces are summarised in Table 1.

Knowing such a kernel representation $\hat{k}_t(\boldsymbol{\nu})$ allows to compute the scale-space image $u(\boldsymbol{x}, t)$ from its Fourier transform

$$\hat{u}(\boldsymbol{\nu}, t) = \hat{k}_t(\boldsymbol{\nu}) \cdot \hat{f}(\boldsymbol{\nu}). \tag{8}$$

3 Morphological Scale-Spaces

Mathematical morphology is based on the concepts of dilation and erosion. The dilation $\oplus$ resp. erosion $\ominus$ of an image f with some structuring function $s : \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty\}$ is defined as

$$(f \oplus s)(\boldsymbol{x}) := \sup_{\boldsymbol{y}\in\mathbb{R}^2} \{f(\boldsymbol{y}) + s(\boldsymbol{x}-\boldsymbol{y})\}, \tag{9}$$

$$(f \ominus s)(\boldsymbol{x}) := \inf_{\boldsymbol{y}\in\mathbb{R}^2} \{f(\boldsymbol{y}) - s(\boldsymbol{y}-\boldsymbol{x})\}. \tag{10}$$

In the following, we only focus on dilation for our derivations.

In order to create scale-space, one performs a so-called *umbral scaling* of the structuring function $s(\boldsymbol{x})$, resulting in

$$s_t(\boldsymbol{x}) := t\, s\left(\frac{\boldsymbol{x}}{t}\right). \tag{11}$$

With $u(.,0) := f$, the (dilation) scale-space evolution $\{u(.,t) \mid t \geq 0\}$ of f is given by

$$u(.,t) = f \oplus s_t. \tag{12}$$

It is possible to derive PDE formulations for such scale-space evolutions, if one considers the *slope transform* of s [15, 7]:

$$\mathcal{S}\,[\,s\,]\,(\boldsymbol{w}) := \operatorname*{stat}_{\boldsymbol{x}\in\mathbb{R}^2} \{s(\boldsymbol{x}) - \langle \boldsymbol{w}, \boldsymbol{x}\rangle\}, \tag{13}$$

where the *stationary values* $\operatorname{stat}_{\boldsymbol{x}} \{h(\boldsymbol{x})\}$ denote the set of function values for which the gradient is zero:

$$\operatorname*{stat}_{\boldsymbol{x}\in\mathbb{R}^2} \{h(\boldsymbol{x})\} := \{h(\boldsymbol{x}) \mid \boldsymbol{\nabla} h(\boldsymbol{x}) = \boldsymbol{0}\}. \tag{14}$$

With these definitions, Dorst and van den Boomgaard [18] have shown that $u(\boldsymbol{x},t)$ from (12) is the solution of

$$\partial_t u = \mathcal{S}\,[\,s\,]\,(\boldsymbol{\nabla} u) \qquad \text{on } \mathbb{R}^2 \times (0,\infty), \tag{15}$$

$$u(\boldsymbol{x},0) = f(\boldsymbol{x}) \qquad \text{on } \mathbb{R}^2. \tag{16}$$

For instance, choosing $s(\boldsymbol{x}) = -\frac{1}{4}|\boldsymbol{x}|^2$ as structuring function gives $\mathcal{S}\,[\,s\,]\,(\boldsymbol{w}) = \boldsymbol{w}^2$. Thus, (15) becomes

$$\partial_t u = |\boldsymbol{\nabla} u|^2. \tag{17}$$

Van den Boomgaard has shown that quadratic structuring functions are the only structuring functions that are rotationally invariant and separable [17]. This has motivated him to regard (17) as the morphological equivalent of the Gaussian scale-space, since the latter one is the only scale-space with a rotationally invariant and separable convolution kernel.
If one uses as structuring function a flat disc

$$s(\boldsymbol{x}) \;=\; \begin{cases} 0 & (|\boldsymbol{x}| \le 1), \\ -\infty & (\text{else}), \end{cases} \tag{18}$$

it has been shown in [2] that one arrives at

$$\partial_t u = |\boldsymbol{\nabla} u|. \tag{19}$$

So far, it was an open question if this equation has a corresponding convolution scale-space. We will answer this later on.
In our following discussion, we will also need the *inverse slope transform*. It is given by (see e.g. [7])

$$\mathcal{S}^{-1}\,[\,h\,]\,(\boldsymbol{x}) = \operatorname*{stat}_{\boldsymbol{y}\in\mathbb{R}^2} \left\{ h(\boldsymbol{y}) + \langle \boldsymbol{x}, \boldsymbol{y} \rangle \right\}. \tag{20}$$

4 Morphological Equivalents of Convolution Scale-Spaces

In order to establish a connection between linear system theory and mathematical morphology, we follow [5]. However, instead of using the *Laplace transform*

$$\mathcal{L}\,[\,f\,]\,(\boldsymbol{x}) = \int_{\mathbb{R}^2} f(\boldsymbol{x}) e^{\langle \boldsymbol{x}, \boldsymbol{y} \rangle}\, d\boldsymbol{y}, \tag{21}$$

we base our computations on the Fourier transform. This has the advantage that we do not require f to decay fast enough at infinity. We introduce a modified version of the Cramér transform which we call *Cramér-Fourier transform*:

$$\mathcal{C}_{\mathcal{F}}\,[\,f\,]\,(\boldsymbol{x}) := \left(-\log(\mathcal{F}\,[\,f\,]\,(\tfrac{\cdot}{2\pi}))\right)^{*}(\boldsymbol{x})\,. \tag{22}$$

Here, $\log \mathcal{F}[f]$ is assumed to be concave, and h^* denotes the *convex conjugate* of a function h:

$$h^*(\boldsymbol{x}^*) := \sup_{\boldsymbol{x}\in\mathbb{R}^2} \{\langle \boldsymbol{x}, \boldsymbol{x}^* \rangle - h(\boldsymbol{x})\}. \tag{23}$$

The key property of the Cramér transform is that it allows to convert a convolution in the usual algebra $(\mathbb{R}, \cdot, +)$ into a dilation in the max-plus algebra $(\mathbb{R} \cup \{-\infty\}, +, \max)$. This was proven by Burgeth and Weickert [5]. In the following lemma we prove the same result for the Cramér-Fourier transform.

Lemma 1. *For two functions f and g with concave logarithmic Fourier transform, it holds*

$$-\mathcal{C}_{\mathcal{F}}[f * g] = (-\mathcal{C}_{\mathcal{F}}[f]) \oplus (-\mathcal{C}_{\mathcal{F}}[g]). \tag{24}$$

Proof. Since convolution in the Fourier domain becomes multiplication, we have

$$\log \mathcal{F}[f * g] = \log(\mathcal{F}[f]\,\mathcal{F}[g]) = \log \mathcal{F}[f] + \log \mathcal{F}[g]\,. \tag{25}$$

Together with a well-known property of convex conjugation (see e.g [16]),

$$(f+g)^*(\boldsymbol{x}) = \inf_{\boldsymbol{y}\in\mathbb{R}^2} (f^*(\boldsymbol{y}) + g^*(\boldsymbol{x}-\boldsymbol{y})), \tag{26}$$

it follows that

$$\begin{aligned}
-\mathcal{C}_{\mathcal{F}}[u * k_t](\boldsymbol{x}) &= -\left(-\log(\mathcal{F}[u](\tfrac{\cdot}{2\pi})) - \log(\mathcal{F}[k_t](\tfrac{\cdot}{2\pi}))\right)^*(\boldsymbol{x}) && (27)\\
&= -\inf_{\boldsymbol{y}\in\mathbb{R}^2} (\mathcal{C}_{\mathcal{F}}[u](\boldsymbol{y}) + \mathcal{C}_{\mathcal{F}}[k_t](\boldsymbol{x}-\boldsymbol{y})) && (28)\\
&= \sup_{\boldsymbol{y}\in\mathbb{R}^2} (-\mathcal{C}_{\mathcal{F}}[u](\boldsymbol{y}) - \mathcal{C}_{\mathcal{F}}[k_t](\boldsymbol{x}-\boldsymbol{y})) && (29)\\
&= (-\mathcal{C}_{\mathcal{F}}[u]) \oplus (-\mathcal{C}_{\mathcal{F}}[k_t])(\boldsymbol{x}). && (30)
\end{aligned}$$

□

In the following we say that a morphological scale-space is *equivalent* to a convolution scale-space, if they result from each other by exchanging the above two algebras. With these definitions and results we can state our main theorem.

Theorem 1. (Morphological Equivalents of Convolution Scale-Spaces). *The morphological equivalents of a convolution scale-space $u(t,\cdot) = f * k_t$ are solutions of*

$$\begin{aligned}
\partial_t u &= \overline{\log \mathcal{F}[k_1]}(\tfrac{1}{2\pi} \boldsymbol{\nabla} u), && (31)\\
u(.,0) &= f, && (32)
\end{aligned}$$

where the bar notation describes $\bar{h}(\boldsymbol{x}) := -h(-\boldsymbol{x})$.

Proof. As a first step we note that for some strictly convex function h we have

$$h^* = \mathcal{S}^{-1}\,[\,-h\,]\,. \tag{33}$$

This can be seen with the definition (20) of the inverse slope transform:

$$\mathcal{S}^{-1}\,[\,-h\,]\,(\boldsymbol{x}) = \operatorname*{stat}_{\boldsymbol{y}\in\mathbb{R}^2}\,\{-h(\boldsymbol{y}) + \langle \boldsymbol{x}, \boldsymbol{y}\rangle\} \tag{34}$$

$$= \sup_{\boldsymbol{y}\in\mathbb{R}^2}\,\{\langle \boldsymbol{x}, \boldsymbol{y}\rangle - h(\boldsymbol{y})\} = h^*(\boldsymbol{x}), \tag{35}$$

since $\langle \boldsymbol{x}, \boldsymbol{y}\rangle - h(\boldsymbol{y})$ is strictly concave in $\boldsymbol{y}$. Therefore, it has a unique stationary value which is a supremum. This proves (33).
With (24), it follows that $-\mathcal{C}_{\mathcal{F}}\,[\,k_t\,]$ creates a morphological scale-space as given in (15)–(16): We obtain

$$\mathcal{S}\,[\,-\mathcal{C}_{\mathcal{F}}\,[\,k_1\,]\,] = \overline{\mathcal{S}\,[\,\mathcal{C}_{\mathcal{F}}\,[\,k_1\,]\,]} \qquad \text{(definition of } \mathcal{S}) \tag{36}$$

$$= \overline{\mathcal{S}\left[\,(-\log\mathcal{F}\,[\,k_1\,]\,(\tfrac{\cdot}{2\pi}))^*\,\right]} \qquad \text{(definition (22))} \tag{37}$$

$$= \overline{\mathcal{S}\left[\,\mathcal{S}^{-1}\left[\,\log\mathcal{F}\,[\,k_1\,]\,(\tfrac{\cdot}{2\pi})\,\right]\,\right]} \qquad \text{(equation (33))} \tag{38}$$

$$= \overline{\log\mathcal{F}\,[\,k_1\,]\,(\tfrac{\cdot}{2\pi})}. \tag{39}$$

This implies the announced equations. □

It should be noted that Theorem 1 is also applicable in those cases where one does not have a closed form representation of the kernel k_t: It is sufficient to know a closed form representation of the Fourier transformed kernel $\hat{k}_t$.

5 Application to Specific Scale-Spaces

Now we are in a position to apply our theory to a number of convolution scale-spaces in order to derive their morphological counterparts.

5.1 Gaussian Scale-Space

Table 1 specifies the Fourier transform of the convolution kernel for Gaussian scale-space as

$$\mathcal{F}\,[\,g_t\,]\,(\boldsymbol{\nu}) = \exp\left(-t\,|2\pi\boldsymbol{\nu}|^2\right). \tag{40}$$

Thus,

$$\mathcal{F}\,[\,g_t\,]\,(\tfrac{1}{2\pi}\boldsymbol{x}) = \exp\left(-t\,|\boldsymbol{x}|^2\right) \tag{41}$$

and Theorem 1 gives the morphological evolution equation

$$\partial_t u = \overline{\log\mathcal{F}\,[\,g_1\,]}(\tfrac{1}{2\pi}\boldsymbol{\nabla} u) = |\boldsymbol{\nabla} u|^2. \tag{42}$$

As expected, this coincides with van den Boomgaard's result [17]. The corresponding structuring function is known to be

$$s_t(\boldsymbol{x}) = -\tfrac{1}{4t}|\boldsymbol{x}|^2. \tag{43}$$

5.2 α-Scale-Spaces

In the same way as above, one can show that the morphological equivalents for the α-scale-spaces are given by

$$\partial_t u = |\boldsymbol{\nabla} u|^{2\alpha}. \tag{44}$$

Interestingly, this proves that for $\alpha = \frac{1}{2}$, the convolution counterpart of the widely-used morphological scale-space

$$\partial_t u = |\boldsymbol{\nabla} u|, \tag{45}$$

which describe dilation with a flat disc of radius t, is given by the Poisson scale-space

$$\partial_t u = -\sqrt{-\Delta}\, u\,. \tag{46}$$

This is a scenario where the morphological process looks simpler and has been discovered nine years before its convolution pendant that involves a pseudo-differential operator [2, 9].
It is also instructive to use our framework for deriving the structuring functions $s_{\alpha,t}$ for the family of morphological α-scale-spaces. Using (15) we know that the dilation α-scale-spaces have to satisfy

$$\mathcal{S}\,[\, s_{\alpha,1}\,]\,(\boldsymbol{\nabla} u) = \partial_t u = |\boldsymbol{\nabla} u|^{2\alpha}. \tag{47}$$

Thus, we can compute s_α with the help of the inverse slope transform and some properties of the convex conjugate (see e.g. [16]):

$$s_{\alpha,t} = \mathcal{S}^{-1}\left[\, t|\boldsymbol{x}|^{2\alpha}\,\right] = -(t|\boldsymbol{x}|^{2\alpha})^* \tag{48}$$

where we have used $\mathcal{S}^{-1}\,[\, h\,] = -h^*$ for a strictly convex h. Since

$$\left(\tfrac{1}{2\alpha}|\boldsymbol{x}|^{2\alpha}\right)^* = \tfrac{2\alpha-1}{2\alpha}|\boldsymbol{x}|^{\frac{2\alpha}{2\alpha-1}}, \tag{49}$$

we get an explicit representation of the structuring function:

$$s_{\alpha,t}(\boldsymbol{x}) = -\left(2\,t\,\alpha\,\tfrac{1}{2\alpha}|\boldsymbol{x}|^{2\alpha}\right)^* = -t\,(2\alpha-1)\left|\frac{\boldsymbol{x}}{t2\alpha}\right|^{\frac{2\alpha}{2\alpha-1}}. \tag{50}$$

Although this formula only holds for $\alpha > \frac{1}{2}$, where strictly concave of the structuring function is guaranteed, we can compute the pointwise limit

$$\lim_{\alpha\to\frac{1}{2}^+} s_{\alpha,t}(\boldsymbol{x}) = \begin{cases} 0 & |\boldsymbol{x}| \le t, \\ -\infty & \text{(else)} \end{cases} \tag{51}$$

and obtain a flat disc of radius t.

5.3 Relativistic Scale-Spaces

From Table 1 we know that

$$\mathcal{F}\left[\, r_{m,t} \,\right](\boldsymbol{\nu}) = \exp\big(-\, t\big(\sqrt{|2\pi\boldsymbol{\nu}|^2 + m^2} - m\big)\big). \tag{52}$$

This gives

$$\overline{\log \mathcal{F}\left[\, r_{m,t} \,\right]}(\tfrac{1}{2\pi}\boldsymbol{x}) = t\big(\sqrt{|\boldsymbol{x}|^2 + m^2} - m\big), \tag{53}$$

and applying Theorem 1 yields the evolution equation of the morphological counterpart for the the relativistic space-spaces:

$$\partial_t u = \sqrt{|\boldsymbol{\nabla} u|^2 + m^2} - m\,. \tag{54}$$

The structuring function can be computed as before as the negative of the convex conjugate of (53):

$$s_{m,t}(\boldsymbol{x}) = -(t\big(\sqrt{|\boldsymbol{x}|^2 + m^2} - m\big))^* \tag{55}$$

$$= -\sup_{\boldsymbol{y}\in\mathbb{R}^2} \Big(-t\big(\sqrt{|\boldsymbol{x}|^2 + m^2} - m\big) + \langle \boldsymbol{x}, \boldsymbol{y}\rangle\Big)\,. \tag{56}$$

With the solution for $\boldsymbol{y}$ given by

$$\boldsymbol{y} = \frac{\boldsymbol{x}\, m}{t^2 - |\boldsymbol{x}|^2} \tag{57}$$

for $|\boldsymbol{x}| \le t$, it follows that

$$s_{m,t}(\boldsymbol{x}) = \begin{cases} m\, t \left(\sqrt{1 - \left(\frac{|\boldsymbol{x}|}{t}\right)^2} - 1 \right) & |\boldsymbol{x}| \le t \\ -\infty & \text{(else).} \end{cases} \tag{58}$$

For $m \to 0$ $s_{m,t}$ converges to a flat disc of radius t. This is expected from the results from the last section since the relativistic scale-spaces converge to the Poisson scale-space for $m \to 0$ and we identified the flat disc of radius t as the structuring function corresponding to the morphological Poisson scale-space. Table 2 summarises the results of this section.

6 Experiments

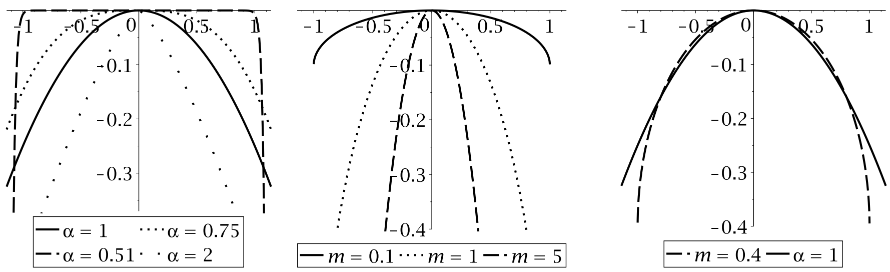
Figure 1 shows a comparison of the structuring functions for all discussed scale-spaces. In these examples, t was chosen to be 1 since umbral scaling can be used to obtain structuring functions for smaller or larger values of t.

To give a visual impression of the morphological counterpart of convolution scale-spaces, Figure 2 and 3 show evolutions of the Mona Lisa image. Whereas the convolution scale-spaces converge to the average greyvalue, their morphological counterparts converge to the brightest greyvalue. For Figure 3, the similarity to a disc-like structuring function is clearly visible for the morphological relativistic scale space with $m = 0.1$.

For the implementation we used a multiplication in the Fourier domain for the convolution scale-spaces. For the morphological scale-spaces, we solved for the maximum over the image domain in Equation (9).

Table 2. Equations for linear scale-spaces and their morphological equivalents.

scale-space	linear (pseudo-)PDE	morphological PDE
Gaussian	$\partial_t u = \Delta u$	$\partial_t u = \lvert\boldsymbol{\nabla} u\rvert^2$
Poisson	$\partial_t u = -\sqrt{-\Delta}\, u$	$\partial_t u = \lvert\boldsymbol{\nabla} u\rvert$
α	$\partial_t u = -(-\Delta)^\alpha u$	$\partial_t u = \lvert\boldsymbol{\nabla} u\rvert^{2\alpha}$
relativistic	$\partial_t u = -\left(\sqrt{-\Delta + m^2} - m\right) u$	$\partial_t u = \sqrt{\lvert\boldsymbol{\nabla} u\rvert^2 + m^2} - m$



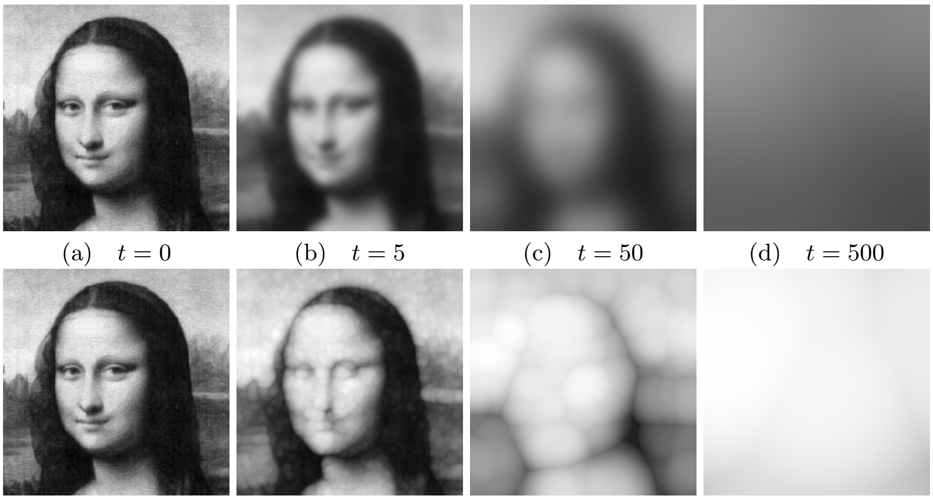
(a) α-scale-space (b) relativistic scale-space (c) comparison of Gaussian and relativistic space-space

Fig. 1. Structuring functions for morphological scale-spaces.

7 Conclusions and Future Work

We have established a mathematical dictionary that allows to translate a convolution scale-space to a morphological scale-space and vice versa. In contrast to previous work on structural similarities between linear and morphological systems, we have achieved these equivalences in the terminology of differential or pseudo-differential operators. We have shown that there exist hitherto unexplored relations between known scale-spaces, such as the Poisson scale-space and morphology with a disc-shaped structuring element. Moreover, we have introduced new morphological scale-spaces that serve as nonlinear counterparts of α-scale-spaces beyond Poisson and Gaussian scale-space, and of relativistic scale-spaces. Their PDE formulations reveal striking structural similarities to their linear pendants.

There are numerous ways to extend these findings in interesting directions. Obviously, these new scale-spaces should be explored further in order to identify promising applications. On the other hand, it is also challenging to generalise this dictionary to other scale-spaces that are not covered within a classical convolution setting, for instance nonlinear diffusion scale-spaces. In this case, Fourier reasonings can no longer be used, and different mathematical techniques are required.



(a) $t = 0$ (b) $t = 5$ (c) $t = 50$ (d) $t = 500$

(e) $t = 0$ (f) $t = 1$ (g) $t = 10$ (h) $t = 100$

Fig. 2. Top: α-scale-space for $\alpha = 0.75$,
Bottom: Morphological α-scale-space for $\alpha = 0.75$

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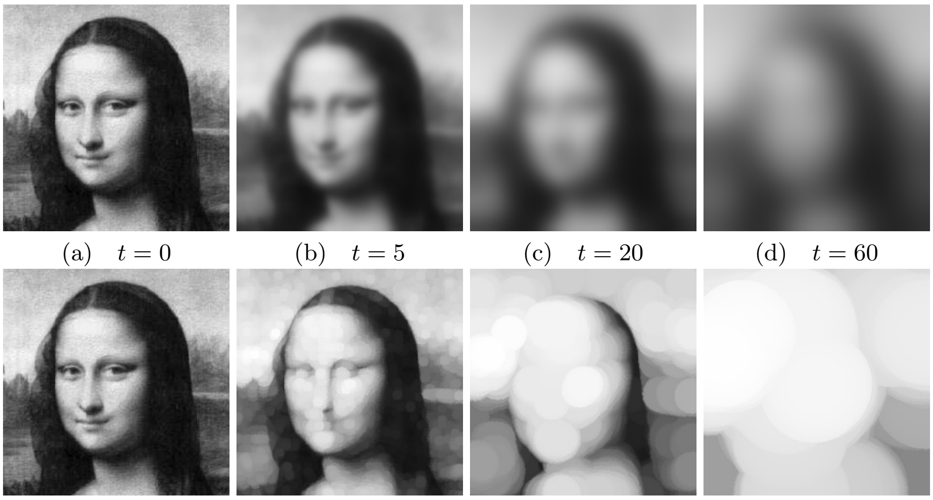

(a) $t = 0$ (b) $t = 5$ (c) $t = 20$ (d) $t = 60$

(e) $t = 0$ (f) $t = 5$ (g) $t = 20$ (h) $t = 60$

Fig. 3. Top: Relativistic scale-space for $m = 0.1$,
Bottom: Morphological Relativistic scale-space for $m = 0.1$

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