Level-Set Methods for Tensor-Valued Images

Christian Feddern Joachim Weickert Bernhard Burgeth

Mathematical Image Analysis Group
Faculty of Mathematics and Computer Science
Saarland University, Building 27
66041 Saarbrücken, Germany
email: {feddern, weickert, burgeth}@mia.uni-saarland.de

Abstract

Tensor-valued data are becoming more and more important as input for todays image analysis problems. This has been caused by a number of applications including diffusion tensor (DT-) MRI and physical measurements of anisotropic behaviour such as stress-strain relationships, interia and permittivity tensors. Consequently, there arises the need to filter and segment such tensor fields. In this paper we extend three important level set methods to tensor-valued data. To this end we first generalise Di Zenzo's concept of a structure tensor for vector-valued images to tensor-valued data. This allows us to derive formulations of mean curvature motion and self-snakes in the case of tensor-valued images. We prove that these processes maintain positive semidefiniteness if the initial matrix data are positive semidefinite. Finally we present a geodesic active contour model for segmenting tensor fields. Since it incorporates information from all channels, it gives a contour representation that is highly robust under noise.

1 Introduction

Starting with Osher and Sethian's pioneering work [28], level set methods have become a fundamental tool in image processing, computer vision and computer graphics [25, 26, 36, 38]. They make use of a number of interesting partial differential equations (PDEs) including mean curvature motion [1], self-snakes [36] and geodesic active contours [7, 22].

While such methods have been extended in various ways to higher dimensions, surfaces and vector-valued data, there are hardly any attempts so far to use them for processing tensor-valued data sets. However, such data sets are becoming increasingly important for three reasons:

1. Novel medical imaging techniques such as diffusion tensor magnetic resonance imaging (DT-MRI) have been introduced [31]. DT-MRI is a 3-D imaging method that yields a diffusion tensor in each voxel. This diffusion tensor describes the diffusive behaviour of water molecules in the tissue. It can be represented by a positive semidefinite 3×3 matrix in each voxel.
2. Tensors have shown their use as a general tool in image analysis, segmentation and grouping [16, 24]. This also includes widespread applications of the so-called structure tensor in fields ranging from motion analysis to texture segmentation; see e.g. [2, 34].
3. A number of scientific applications require the visualisation and processing of tensor fields [39]. The tensor concept is a common physical description of anisotropic behaviour, especially in solid mechanics and civil engineering (e.g. stress-strain relationships, inertia tensors, diffusion tensors, permittivity tensors).

The search for good smoothing techniques for DT-MRI data and related tensor fields is a very recent research area. Several authors have addressed this problem by smoothing derived expressions such as the eigenvalues and eigenvectors of the diffusion tensor [12, 32, 40] or rotationally invariant scalar-valued expressions [29, 47]. Also for fiber tracking applications, most techniques work on scalar- or vector-valued data [5, 42]. Some image processing methods that work directly on the tensor components use linear [45] or nonlinear [19] techniques that filter all channels *independently*, thus performing scalar-valued filtering again. Nonlinear variational methods for matrix-valued filtering with channel coupling have been proposed both in the isotropic [40] and in the anisotropic setting [44]. Related nonlinear diffusion methods for tensor-valued data have led to the notion of a nonlinear structure tensor [44] that has been used for optic flow estimation [4], texture discrimination and tracking [3]. To the best of our knowledge, however, no level-set methods have been proposed so far that work directly on the tensor data.

The goal of the present paper is to introduce three level-set methods for analysing and processing tensor fields. They can be regarded as tensor-valued extensions of mean curvature motion, self-snakes and geodesic active contours. The key ingredient for this generalisation is the use of a structure tensor for matrix-valued data. It may be regarded as a generalisation of Di Zenzo's edge detector for vector-valued images [13]. For simplicity, we restrict ourselves to the case of 2×2 matrix fields, but most of the concepts can also be generalised to higher dimensions.

Our paper is organised as follows. In Section 2 we introduce the generalised structure tensor for matrix fields. It is then used in Section 3 for designing a mean curvature type evolution of tensor-valued data. We prove that this process preserves the positive semidefiniteness of the initial data. Modifying tensor-valued mean curvature motion by a suitable edge stopping function leads us to tensor-valued self-snakes. They are discussed in Section 4. In Section 5, we use the self-snake model in order to derive geodesic active contour models for tensor fields. Algorithmic details are sketched in Section 6, and experiments are presented in Section 7. The paper is concluded with a summary in Section 8.

2 Structure Analysis of Tensor-Valued Data

In this section we generalise the concept of an image gradient to the tensor-valued setting. This may be regarded as a tensor extension of Di Zenzo's method for vector-valued data [13].

Let us consider some 2×2 tensor image $(f_{i,j}(x,y))$ where the indices (i,j) specify the tensor channel. We would like to define an "edge direction" for such a matrix-valued function. In the case of some scalar-valued image $f(x,y)$, we would look for the direction v which is orthogonal to the gradient of a Gaussian-smoothed version of f:

$$0 = v^\top \nabla f_\sigma \tag{1}$$

where $f_\sigma := K_\sigma * f$ and K_σ denotes a Gaussian with standard deviation σ. Gaussian convolution makes the structure detection more robust against noise. The parameter σ is called *noise scale*.
In the general tensor-valued case, we cannot expect that all tensor channels yield the same edge direction. Therefore we proceed as follows. Let $f_{\sigma,i,j}$ be a Gaussian-smoothed version of $f_{i,j}$. Then we define the edge direction as the unit vector v that minimises

$$\begin{aligned} E(v) &:= \sum_{i=1}^{2}\sum_{j=1}^{2}(v^\top \nabla f_{\sigma,i,j})^2 \\ &= v^\top \Big(\sum_{i=1}^{2}\sum_{j=1}^{2} \nabla f_{\sigma,i,j} \nabla f_{\sigma,i,j}^\top\Big) v. \end{aligned} \tag{2}$$

This quadratic form is minimised, when v is eigenvector to the smallest eigenvalue of the *structure tensor*

$$J(\nabla f_\sigma) := \sum_{i=1}^{2}\sum_{j=1}^{2} \nabla f_{\sigma,i,j} \nabla f_{\sigma,i,j}^\top. \tag{3}$$

The trace of this matrix can be regarded as a tensor-valued generalisation of the squared gradient magnitude:

$$\mathrm{tr} J(\nabla f_\sigma) = \sum_{i=1}^{2}\sum_{j=1}^{2} |\nabla f_{\sigma,i,j}|^2. \tag{4}$$

The matrix $J(\nabla f_\sigma)$ will allow us to generalise a number of level-set methods to the tensor-valued setting. Indeed, extending the ideas in [11] to the matrix-valued case, one my even define level sets of matrix-valued as the integral curves of the eigenvector directions to the smallest eigenvalue.

3 Mean Curvature Motion

In this section we introduce a tensor-valued mean curvature motion. To this end, we first have to sketch some basic ideas behind scalar-valued mean curvature motion.

We start with the observation that the Laplacian of an isotropic linear diffusion model may be decomposed into two orthogonal directions $\xi \perp \nabla u$ and $\eta \parallel \nabla u$:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + \partial_{yy} u & (5)\\ &= \partial_{\xi\xi} u + \partial_{\eta\eta} u & (6) \end{aligned}$$

where $\partial_{\xi\xi} u$ describes smoothing parallel to edges and $\partial_{\eta\eta}$ smoothes perpendicular to edges. Mean curvature motion (MCM) uses an anisotropic variant of this smoothing process by permitting only smoothing along the level lines:

$$\partial_t u = \partial_{\xi\xi} u \tag{7}$$

This can be rewritten as

$$\partial_t u = |\nabla u| \text{ div } \left(\frac{\nabla u}{|\nabla u|} \right). \tag{8}$$

Alvarez *et al.* have used this evolution equation for denoising highly degraded images [1]. It is well-known from the mathematical literature [14, 17, 20] that under MCM convex level lines remain convex, nonconvex ones become convex, and in finite time they vanish by shrinking to circular points (a point with a circle as limiting shape). Interestingly, mean curvature motion plays a similar role for morphology as linear diffusion does in the context of linear averaging. While iterated and suitably scaled convolutions with smoothing masks approximate linear diffusion filtering, it has been shown that iterated classic morphological operators such as median filtering are approximating MCM [18]. Similar approximation results can also be established in the vector-valued setting [8].

If we want to use a MCM-like process for processing tensor-valued data $(f_{i,j})$, it is natural to replace the second directional derivative $\partial_{\xi\xi} u$ in (7) by $\partial_{vv} u$, where v is the eigenvector to the smallest eigenvalue of the structure tensor $J(\nabla u)$. This leads us to the evolution

$$\begin{aligned} \partial_t u_{i,j} &= \partial_{vv} u_{i,j} & (9)\\ u_{i,j}(x,y,0) &= f_{i,j}(x,y) & (10) \end{aligned}$$

for all tensor channels (i,j). Note that this process synchronises the smoothing direction in all channels. It may be regarded as a tensor-valued generalisation of the vector-valued mean curvature motion proposed by Chambolle [10] and its modifications by Sapiro and Ringach [37]. The synchronisation of channel smoothing is also a frequently used strategy in vector-valued diffusion filtering [15, 43, 41].

Preservation of Positive Semidefiniteness of the Tensor Field under Mean Curvature Motion. A number of tensor-valued data sets reveal additional properties such as positive semidefiniteness. Hence it would be desirable that an image processing method does not destroy this property. Let us now prove that twice differentiable solutions of MCM on the unbounded domain $\mathbb{R}^2 \times]0,+\infty[$ do preserve the positive semidefiniteness of the initial data. We are optimistic that the reasoning below can also be extended to more general solution concepts such as viscosity solutions.

The tensor field $U(x,y,t) = (u_{i,j}(x,y,t))$ satisfying the evolution equation (9) is associated with the scalar-valued function representing the smaller eigenvalue $\lambda_{min}(x,y,t)$ of the matrix $U(x,y,t)$ at the point (x,y,t) and the well-known Rayleigh-quotient

$$u^d(x,y,t) \quad = \quad d^\top U(x,y,t)\, d, \qquad (11)$$

where $d \in \mathbb{R}^2$ with $\|d\| = 1$. Let (x_0,y_0,τ) be a local minimum of the function λ_{min}. Choosing d as a suitable eigenvector we obtain

$$u^d(x_0,y_0,\tau) = \lambda_{min}(x_0,y_0,\tau). \qquad (12)$$

Assume in equation (7) that $\xi \in \mathbb{R}^2$ is equal to the normalised vector v referred to in (9) as the eigenvector of the structure tensor $J(\nabla u)$ at the minimum point (x_0,y_0,τ), that is, the direction of the isoline of the tensor field. Thanks to the linearity of the differential operators involved, u^d satisfies

$$(u^d)_t = (u^d)_{\xi\xi} \text{ for any } d \in \mathbb{R}^2. \qquad (13)$$

A minimum is always a point of convexity, which implies for twice differentiable functions w that $\mathrm{Hess}(w(x_0,y_0,\tau))$ is positive semidefinite. Hence in view of (12), (13) we have

$$\begin{aligned} \partial_t \lambda_{min}(x_0,y_0,\tau) &= \partial_{\xi\xi}\lambda_{min}(x_0,y_0,\tau) \\ &= \xi^\top \mathrm{Hess}(\lambda_{min}(x_0,y_0,\tau))\,\xi \\ &\geq 0 \end{aligned} \qquad (14)$$

if ξ represents the direction of the isoline of the corresponding tensor field at the point (x_0,y_0,τ). In other words: at a minimum point the smallest eigenvalue λ_{min} of the matrix in that point is *increasing in time*. This in turn implies preservation of positivity of the smallest eigenvalue. Hence the positive semidefiniteness of the initial tensor field is maintained.

4 Self-Snakes

In [35], Sapiro has proposed a specific variant of MCM that is well-suited for image enhancement. This process which he names *self-snakes* introduces an edge-stopping function into mean curvature motion in order to prevent further shrinkage of the level lines once they have reached important image edges. In the scalar-valued setting, a self-snake $u(x,y,t)$ of some image $f(x,y)$ is generated by the evolution process

$$\partial_t u \quad = \quad |\nabla u| \,\mathrm{div}\left(g(|\nabla u|^2)\,\frac{\nabla u}{|\nabla u|}\right), \qquad (15)$$

$$u(x,y,0) \quad = \quad f(x,y), \qquad (16)$$

where g is a decreasing function such as the Perona-Malik diffusivity [30]

$$g(|\nabla u|^2) := \frac{1}{1+|\nabla u|^2/\lambda^2}. \qquad (17)$$

In order to make self-snakes more robust under noise it is common to replace $g(|\nabla u|^2)$ by its Gaussian-regularised variant $g(|\nabla u_\sigma|^2)$. Self-snakes have been advocated as alternatives to nonlinear diffusion filters [46], they can be used for vector-valued images [35], and related processes have also been proposed for filtering 3-D images [33].

Using the product rule of differentiation, we may rewrite Equation (15) as

$$\partial_t u \;=\; g(|\nabla u_\sigma|^2)\,\partial_{\xi\xi} u + \nabla^\top(g(|\nabla u_\sigma|^2))\,\nabla u. \qquad (18)$$

This formulation suggests a straightforward generalisation to the tensor-valued setting. All we have to do is to replace $|\nabla u_\sigma|^2$ by $\mathrm{tr}J(\nabla u_\sigma)$, and $\partial_{\xi\xi}$ by ∂_{vv} where v is the eigenvector to the smallest eigenvalue of $J(\nabla u)$. This lead us to the following tensor-valued evolution:

$$\begin{aligned} \partial_t u_{i,j} &= g(\mathrm{tr}J(\nabla u_\sigma))\,\partial_{vv} u_{i,j} \\ &\quad + \nabla^\top(g(\mathrm{tr}J(\nabla u_\sigma)))\,\nabla u_{i,j}, \end{aligned} \qquad (19)$$

$$u_{i,j}(x,y,0) \quad = \quad f_{i,j}(x,y). \qquad (20)$$

We observe that the main difference to tensor-valued MCM consists of the additional term $\nabla^\top(g(\mathrm{tr}J(\nabla u_\sigma)))\,\nabla u_{i,j}$. It can be regarded as a shock term [27] that is responsible for the edge-enhancing properties of self-snakes.

With only minor modifications, it is possible to extend the semidefiniteness preservation proof for tensor-valued MCM also to the case of tensor-valued self-snakes.

5 Active Contour Models

Active contours [21] play an important role in interactive image segmentation, in particular for medical applications. The underlying idea is that the user specifies an initial guess of an interesting contour (organ, tumour, ...). Then this contour is moved by image-driven forces to the edges of the object in question.

So-called geodesic active contour models [7, 22] achieve this by applying a specific kind of level set ideas. They may be regarded as extensions of the implicit snake models in [6, 23]. In its simplest form, a geodesic active contour model consists of the following steps. One embeds the user-specified initial curve $C_0(s)$ as a zero level curve into a function $f(x,y)$, for instance by using the distance transformation. Then f is evolved under a PDE which includes knowledge about the original image h:

$$\partial_t u = |\nabla u| \operatorname{div} \left(g(|\nabla h_\sigma|^2) \frac{\nabla u}{|\nabla u|} \right), \quad (21)$$

$$u(x,y,0) = f(x,y), \quad (22)$$

where g inhibits evolution at edges of f. One may choose decreasing functions such as the Perona–Malik diffusivity (17). Experiments indicate that, in general, (21) will have nontrivial steady states. The evolution is stopped at some time T, when the process does hardly alter anymore, and the final contour C is extracted as the zero level curve of $u(x,T)$.

To extend this idea to tensor valued data $h_{i,j}$, we propose to use $\text{tr}(J(\nabla h_\sigma))$ as argument of the stopping function g.

$$\partial_t u = |\nabla u| \operatorname{div} \left(g(\text{tr}J(\nabla h_\sigma)) \frac{\nabla u}{|\nabla u|} \right). \quad (23)$$

Note that, in contrast to the processes in the previous section, this equation is still scalar-valued, since the goal is to find a contour that segments all channels simultaneously. The active contour evolution for this process may be rewritten as

$$\begin{aligned} \partial_t u &= g(\text{tr}J(\nabla h_\sigma))\, \partial_{\xi\xi} u \\ &\quad + \nabla^\top (g(\text{tr}J(\nabla h_\sigma)))\, \nabla u, \end{aligned} \quad (24)$$

$$u(x,y,0) = f(x,y). \quad (25)$$

Since a tensor-valued image involves more channels than a scalar-valued one, we can expect that this additional information stabilises the process when noise is present. Our experiments in Section 7 will confirm this expectation.

6 Numerical Approaches

Our implementation is based on explicit finite difference schemes for the evolutions (9), (19) and (24). For computing the structure tensor of a matrix field, we replaced the derivatives by central differences. Gaussian convolution was performed in the spatial domain with a sampled renormalised Gaussian K_σ that has been truncated at $\pm 3\sigma$. Its symmetry and separability has been used to accelerate the convolution. Since the structure tensor is a 2×2 matrix, one can compute its eigenvectors analytically.

The time derivative in the evolution PDEs has been replaced by a forward difference, and discretisations of second-order directional derivatives are based on the formula

$$\partial_{vv} u = c^2 \partial_{xx} u + 2cs \partial_{xy} u + s^2 \partial_{yy} u, \quad (26)$$

where $v = (c,s)^2$ denotes a unit vector. Spatial derivatives $\partial_{xx} u$, $\partial_{xy} u$ and $\partial_{yy} u$ are approximated by central differences.

The shock terms of type $\nabla^\top g \, \nabla u$ involve first order spatial derivatives. In this case we have used central differences for approximating ∇g and upwind discretisations for ∇u. For more details on upwind schemes for level set ideas we refer to [28].

Our experiments have shown that the resulting explicit schemes are stable for time step sizes $\tau \le 0.25$ when the pixel size is set to 1.

7 Experiments

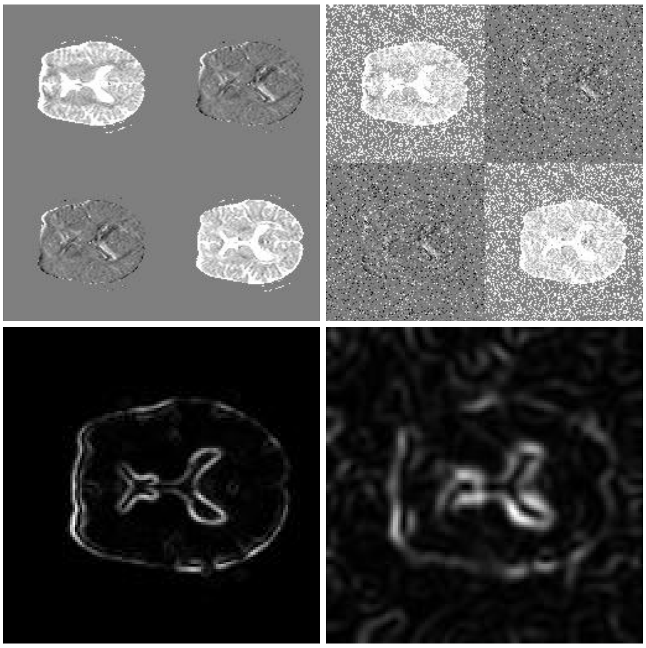

Figure 1: Edge detection with a structure tensor for matrix-valued data. (a) *Top left:* Original 2-D tensor field extracted from a 3-D DT-MRI data set by using the channels $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$. Each channel is of size 256×256. The channels $(1,2)$ and $(2,1)$ are identical for symmetry reasons. (b) *Top right:* Same image with 30 % noise. (c) *Bottom left:* Trace of the structure tensor of the original data. ($\sigma = 1$). (d) *Bottom right:* Trace of the structure tensor from the noisy image ($\sigma = 3$).

The test image we used for our experiments was obtained from an MRI data set of a human brain. We have extracted a 2-D section from the 3-D data. The 2-D image consists of four quadrants which show the four tensor channels of a 2×2 matrix. Each channel has a resolution of 256×256 pixels. The top right channel and bottom left channel are identical since the matrix is symmetric. To test the robustness under noise we have replaced 30 % of all data by noise matrices: The angle of their eigensystem was uniformly distributed in $[0, \pi]$, and their eigenvalues are uniformly distributed in the range $[0, 127]$. We applied all our methods to both the original image and the noisy image.

Figure 1 demonstrates the use of $\text{tr}J(\nabla f_\sigma)$ for detecting edges in tensor-valued images. We observe that this method gives good results for the original data set. When increas-

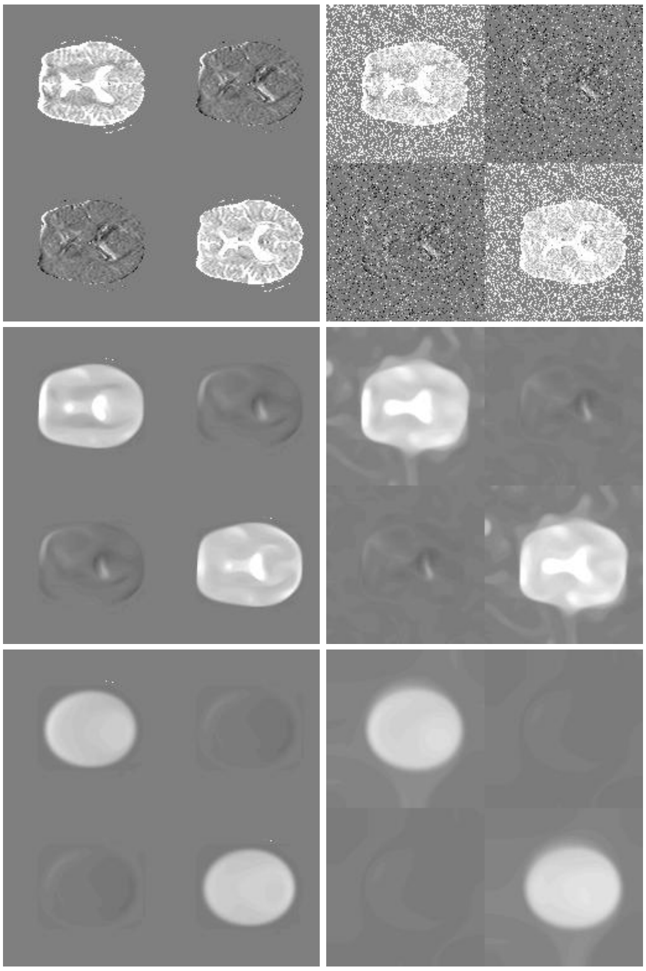

Figure 2: Tensor-valued mean curvature motion. *Left column, from top to bottom:* Original tensor image of size 256×256, at time $t = 24$, at time $t = 240$. *Right column, from top to bottom:* Same experiment with 30 % noise.

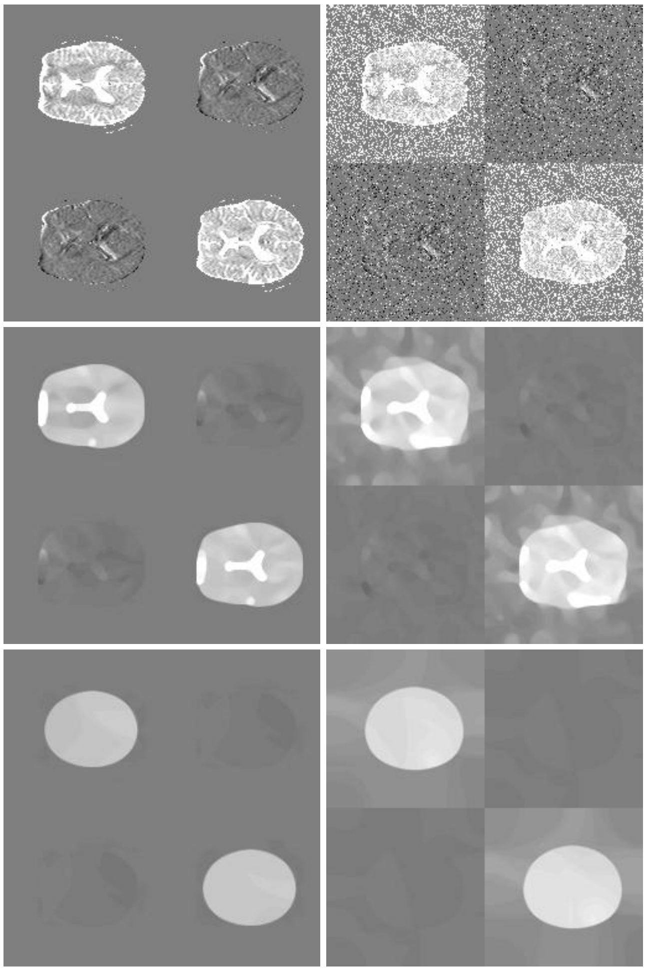

Figure 3: Tensor-valued self-snakes ($\sigma = 2$, $\lambda = 2$). *Left column, from top to bottom:* Original tensor image of size 256×256, at time $t = 24$, at time $t = 240$. *Right column, from top to bottom:* Same experiment with 30 % noise.

ing the noise scale σ, it is also possible to handle situations where massive noise is present.

In our next experiment we applied the tensor-valued mean curvature model to the test images. As can be seen in the first column of Figure (2), the results look very similar to evolutions that one experiences with scalar-valued mean curvature motion: Level lines become convex and shrink towards circles before they vanish in finite time. This indicates that our method is a good extension to tensor-valued data. The second column of Figure (2) shows the same algorithm applied to the noisy image. As expected it possesses a high robustness to noise. Large evolution times give nearly identical results for the original and the noisy images.

The results for the tensor-valued self-snake algorithm are shown in Figure 3. They look similar to the MCM results, but they offer better sharpness at edges due to the additional shock term. On the other hand, noise is more noticeable than with mean curvature motion. This effect resembles the difference between linear and nonlinear diffusion filters. The nonlinear diffusivity that is responsible for sharp edges may also misinterpret high-gradient noise as an edge that deserves to be preserved. This drawback may be circumvented by choosing a sufficiently large noise scale σ; see also [9] for related ideas in the context of diffusion filtering.

Finally, we applied our active contour model to the same tensor data sets. The goal was to extract the contour of the human brain shown on the original image. Figure 4 shows the temporal evolution of the active contours. First one notices that the evolution is slower in the noisy case. This is caused by the fact that noise creates large values in the trace of the structure tensor. This in turn slows down the evolution. For larger times, however, both results become very similar. This shows the high noise robustness of our active contour model for tensor-valued data sets. A comparison with an uncoupled active contour model in the bottom row of Figure 4 illustrates the superiority of the proposed channel coupling.

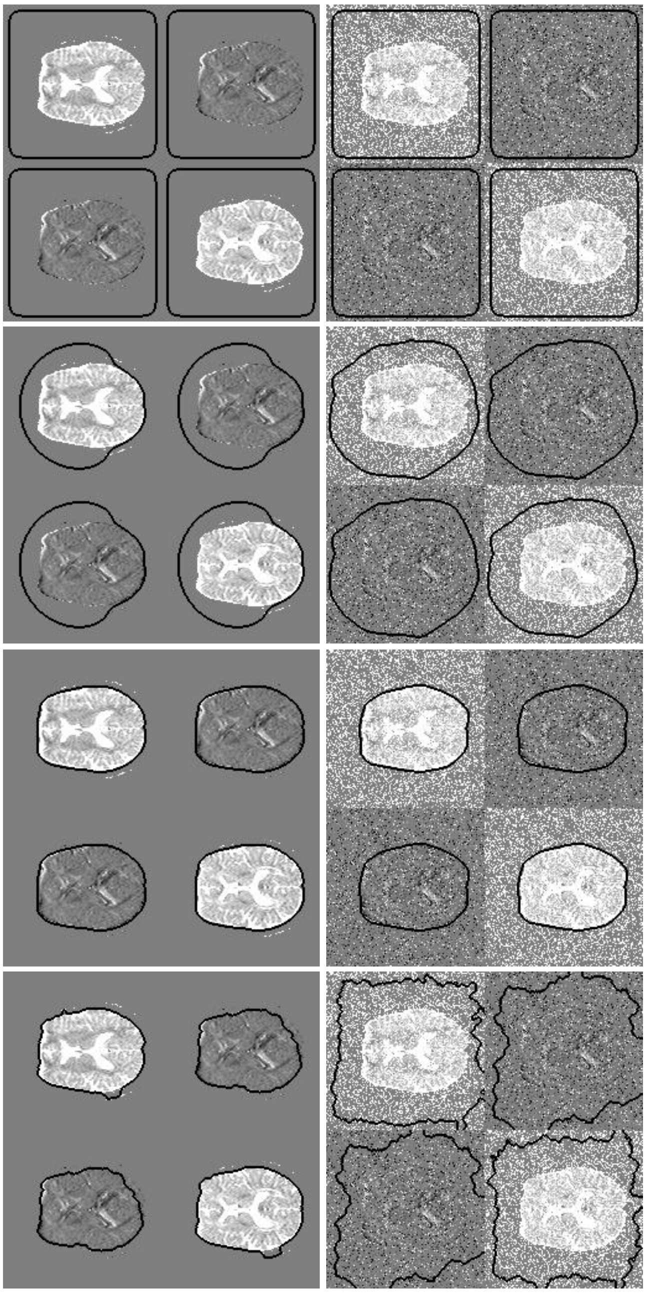

Figure 4: Tensor-valued geodesic active contours ($\sigma = 3$, $\lambda = 1$). *Left column, from top to 3rd line:* Tensor image of size 256×256 including contour at time $t = 0$, $t = 960$ and $t = 9600$. *Right column, from top to 3rd line:* Same experiment with 30 % noise. *Bottom left:* Uncoupled active contours for the original image, $t = 9600$. *Bottom right:* Ditto for the noisy image.

8 Summary and Conclusions

In this paper we have described how the scalar-valued level set methods based on mean curvature motion, self-snakes and geodesic active contour models can be extended to tensor-valued data. We have shown that evolutions under tensor-valued mean curvature motion or tensor-valued self-snakes give positive semidefinite results for all positive semidefinite initial tensor fields. Experiments illustrate that the proposed tensor-valued methods inherit characteristic properties of their scalar-valued counterparts, but, by using tensor-valued input data, they robustness under noise improves. This is a result from the fact that all tensor channels simultaneously contribute to the calculation of the structure tensor that steers the process. We are currently extending our 2-D implementations to 3-D data sets and we are investigating efficient numerical schemes for this purpose.

Acknowledgments

We are grateful to Anna Vilanova i Bartrolí (Eindhoven Institute of Technology) and Carola van Pul (Maxima Medical Center, Eindhoven) for providing us with the DT-MRI dataset and for discussing questions concerning data conversion. Moreover, we would like to thank Martin Welk (Saarland University) for helping us with the experiments.

References

[1] L. Alvarez, P.-L. Lions, and J.-M. Morel. Image selective smoothing and edge detection by nonlinear diffusion. II. *SIAM Journal on Numerical Analysis*, 29:845–866, 1992.

[2] J. Bigün, G. H. Granlund, and J. Wiklund. Multidimensional orientation estimation with applications to texture analysis and optical flow. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 13(8):775–790, August 1991.

[3] T. Brox, M. Rousson, R. Deriche, and J. Weickert. Unsupervised segmentation incorporating colour, texture, and motion. In N. Petkov, editor, *Computer Analysis of Images and Patterns*, Lecture Notes in Computer Science. Springer, Berlin, 2003. In press.

[4] T. Brox and J. Weickert. Nonlinear matrix diffusion for optic flow estimation. In L. Van Gool, editor, *Pattern Recognition*, volume 2449 of *Lecture Notes in Computer Science*, pages 446–453. Springer, Berlin, 2002.

[5] J. Campbell, K. Siddiqi, B. Vemuri, and G. B. Pike. A geometric flow for white matter fibre tract reconstruction. In *Proc. 2002 IEEE International Symposium on Biomedical Imaging*, pages 505–508, Washington, DC, July 2002.

[6] V. Caselles, F. Catté, T. Coll, and F. Dibos. A geometric model for active contours in image processing. *Numerische Mathematik*, 66:1–31, 1993.

[7] V. Caselles, R. Kimmel, and G. Sapiro. Geodesic active contours. In *Proc. Fifth International Conference on Computer Vision*, pages 694–699, Cambridge, MA, June 1995. IEEE Computer Society Press.

[8] V. Caselles, G. Sapiro, and D. H. Chung. Vector median filters, inf-sup convolutions, and coupled PDE's: theoretical connections. *Journal of Mathematical Imaging and Vision*, 12(2):109–119, April 2000.

[9] F. Catté, P.-L. Lions, J.-M. Morel, and T. Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM Journal on Numerical Analysis*, 32:1895–1909, 1992.

[10] A. Chambolle. Partial differential equations and image processing. In *Proc. 1994 IEEE International Conference on Image Processing*, volume 1, pages 16–20, Austin, TX, November 1994. IEEE Computer Society Press.

[11] D. H. Chung and G. Sapiro. On the level lines and geometry of vector-valued images. *IEEE Signal Processing Letters*, 7(9):241–243, 2000.

[12] O. Coulon, D. C. Alexander, and S. A. Arridge. A regularization scheme for diffusion tensor magnetic resonance images. In M. F. Insana and R. M. Leahy, editors, *Information Processing in Medical Imaging – IPMI 2001*, volume 2082 of *Lecture Notes in Computer Science*, pages 92–105. Springer, Berlin, 2001.

[13] S. Di Zenzo. A note on the gradient of a multi-image. *Computer Vision, Graphics and Image Processing*, 33:116–125, 1986.

[14] M. Gage and R. S. Hamilton. The heat equation shrinking convex plane curves. *Journal of Differential Geometry*, 23:69–96, 1986.

[15] G. Gerig, O. Kübler, R. Kikinis, and F. A. Jolesz. Nonlinear anisotropic filtering of MRI data. *IEEE Transactions on Medical Imaging*, 11:221–232, 1992.

[16] G. H. Granlund and H. Knutsson. *Signal Processing for Computer Vision*. Kluwer, Dordrecht, 1995.

[17] M. Grayson. The heat equation shrinks embedded plane curves to round points. *Journal of Differential Geometry*, 26:285–314, 1987.

[18] F. Guichard and J.-M. Morel. Partial differential equations and image iterative filtering. In I. S. Duff and G. A. Watson, editors, *The State of the Art in Numerical Analysis*, number 63 in IMA Conference Series (New Series), pages 525–562. Clarendon Press, Oxford, 1997.

[19] K. Hahn, S. Pigarin, and B. Pütz. Edge preserving regularization and tracking for diffusion tensor imaging. In W. J. Niessen and M. A. Viergever, editors, *Medical Image Computing and Computer-Assisted Intervention – MICCAI 2001*, volume 2208 of *Lecture Notes in Computer Science*, pages 195–203. Springer, Berlin, 2001.

[20] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *Journal of Differential Geometry*, 20:237–266, 1984.

[21] M. Kass, A. Witkin, and D. Terzopoulos. Snakes: Active contour models. *International Journal of Computer Vision*, 1:321–331, 1988.

[22] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, and A. Yezzi. Gradient flows and geometric active contour models. In *Proc. Fifth International Conference on Computer Vision*, pages 810–815, Cambridge, MA, June 1995. IEEE Computer Society Press.

[23] R. Malladi, J. A. Sethian, and B. C. Vemuri. A topology independent shape modeling scheme. In B. Vemuri, editor, *Geometric Methods in Computer Vision*, volume 2031 of *Proceedings of SPIE*, pages 246–258. SPIE Press, Bellingham, 1993.

[24] G. Medioni, M.-S. Lee, and C.-K. Tang. *A Computational Framework for Segmentation and Grouping*. Elsevier, Amsterdam, 2000.

[25] S. Osher and R. P. Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces*, volume 153 of *Applied Mathematical Sciences*. Springer, New York, 2002.

[26] S. Osher and N. Paragios, editors. *Geometric Level Set Methods in Imaging, Vision and Graphics*. Springer, New York, 2003.

[27] S. Osher and L. I. Rudin. Feature-oriented image enhancement using shock filters. *SIAM Journal on Numerical Analysis*, 27:919–940, 1990.

[28] S. Osher and J. A. Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton–Jacobi formulations. *Journal of Computational Physics*, 79:12–49, 1988.

[29] G. J. M. Parker, J. A. Schnabel, M. R. Symms, D. J. Werring, and G. J. Barker. Nonlinear smoothing for reduction of systematic and random errors in diffusion tensor imaging. *Journal of Magnetic Resonance Imaging*, 11:702–710, 2000.

[30] P. Perona and J. Malik. Scale space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12:629–639, 1990.

[31] C. Pierpaoli, P. Jezzard, P. J. Basser, A. Barnett, and G. Di Chiro. Diffusion tensor MR imaging of the human brain. *Radiology*, 201(3):637–648, December 1996.

[32] C. Poupon, J.-F. Mangin, V. Frouin, J. Régis, F. Poupon, M. Pachot-Clouard, D. Le Bihan, and I. Bloch. Regularization of MR diffusion tensor maps for tracking brain white matter bundles. In W. M. Wells, A. Colchester, and S. Delp, editors, *Medical Image Computing and Computer-Assisted Intervention – MICCAI 1998*, volume 1496 of *Lecture Notes in Computer Science*, pages 489–498. Springer, Berlin, 1998.

[33] T. Preußer and M. Rumpf. A level set method for anisotropic geometric diffusion in 3D image processing. *SIAM Journal on Applied Mathematics*, 62(5):1772–1793, 2002.

[34] A. R. Rao and B. G. Schunck. Computing oriented texture fields. *CVGIP: Graphical Models and Image Processing*, 53:157–185, 1991.

[35] G. Sapiro. Vector (self) snakes: a geometric framework for color, texture and multiscale image segmentation. In *Proc. 1996 IEEE International Conference on Image Processing*, volume 1, pages 817–820, Lausanne, Switzerland, September 1996.

[36] G. Sapiro. *Geometric Partial Differential Equations and Image Analysis*. Cambridge University Press, Cambridge, UK, 2001.

[37] G. Sapiro and D. L. Ringach. Anisotropic diffusion of multivalued images with applications to color filtering. *IEEE Transactions on Image Processing*, 5(11):1582–1586, 1996.

[38] J. A. Sethian. *Level Set Methods and Fast Marching Methods*. Cambridge University Press, Cambridge, UK, second edition, 1999.

[39] X. Tricoche, G. Scheuermann, and H. Hagen. Vector and tensor field topology simplification on irregular grids. In D. Ebert, J. M. Favre, and R. Peikert, editors, *Proc. Joint Eurographics – IEEE TCVG Symposium on Visualization*, pages 107–116. Springer, Wien, 2001.

[40] D. Tschumperlé and R. Deriche. Orthonormal vector sets regularization with PDE's and applications. *International Journal of Computer Vision*, 50(3):237–252, December 2002.

[41] D. Tschumperlé and R. Deriche. Vector-valued image regularization with PDE's: a common framework for different applications. In *Proc. 2003 IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, Madison, WI, June 2003. IEEE Computer Society Press.

[42] B. Vemuri, Y. Chen, M. Rao, T. McGraw, Z. Wang, and T. Mareci. Fiber tract mapping from diffusion tensor MRI. In *Proc. First IEEE Workshop on Variational and Level Set Methods in Computer Vision*, pages 73–80, Vancouver, Canada, July 2001. IEEE Computer Society Press.

[43] J. Weickert. Coherence-enhancing diffusion of colour images. *Image and Vision Computing*, 17(3–4):199–210, March 1999.

[44] J. Weickert and T. Brox. Diffusion and regularization of vector- and matrix-valued images. In M. Z. Nashed and O. Scherzer, editors, *Inverse Problems, Image Analysis, and Medical Imaging*, volume 313 of *Contemporary Mathematics*, pages 251–268. AMS, Providence, 2002.

[45] C.-F. Westin, S. E. Maier, B. Khidhir, P. Everett, F. A. Jolesz, and R. Kikinis. Image processing for diffusion tensor magnetic resonance imaging. In C. Taylor and A. Colchester, editors, *Medical Image Computing and Computer-Assisted Intervention – MICCAI 1999*, volume 1679 of *Lecture Notes in Computer Science*, pages 441–452. Springer, Berlin, 1999.

[46] R. T. Whitaker and X. Xue. Variable-conductance, level-set curvature for image denoising. In *Proc. 2001 IEEE International Conference on Image Processing*, pages 142–145, Thessaloniki, Greece, October 2001.

[47] L. Zhukov, K. Munseth, D. Breen, R. Whitaker, and A. H. Barr. Level set modeling and segmentation of DT-MRI brain data. *Journal of Electronic Imaging*, 2003. To appear.