Stability and Local Feature Enhancement of Higher Order Nonlinear Diffusion Filtering

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Abstract. This paper discusses the extension of nonlinear diffusion filters to higher derivative orders. While such processes can be useful in practice, their theoretical properties are only partly understood so far. We establish important results concerning L^2-stability and forward-backward diffusion properties which are related to well-posedness questions. Stability in the L^2-norm is proven for nonlinear diffusion filtering of arbitrary order. In the case of fourth order filtering, a qualitative description of the filtering behaviour in terms of forward and backward diffusion is given and compared to second order nonlinear diffusion. This description shows that *curvature enhancement* is possible with of fourth order nonlinear diffusion in contrast to second order filters where only edges can be enhanced.

1 Introduction

Nonlinear diffusion filtering is an established method for signal and image denoising and simplification. Starting with the pioneering work of Perona and Malik [1] in 1990, investigations have covered both the theoretical properties and the usefulness in practice of nonlinear diffusion filters and related variational methods [2,3,4,5,6,7]. With a whole spectrum of different diffusivities, the method produces smoothing effects as well as edge enhancement. The edge enhancement locally increases the first derivative and may also cause one of the major drawbacks of the method, the so-called *staircasing effect*: Regions with smooth grey value changes in the original signal or image can be turned into many segmentation-like regions. Fig. 1 (b.) shows a denoising example where this effect occurs. To circumvent these artifacts, higher derivative orders have been introduced in the diffusion process [8,9,10,11,12,13]. This allows a higher adaptivity to the local image structure and can yield piecewise linear regions. Several methods have been proposed which confirm the impression that higher order nonlinear diffusion is a very useful extension of the well-established second order methods in practice, see also Fig. 1 (c.). Unfortunately, not many theoretical properties have been established for higher-order smoothing so far.

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Fig. 1. (a.) *Left:* Initial noisy image. (b.) *Middle:* Second order Perona-Malik filtering. (c.) *Right:* Linear combination of second and fourth order Perona-Malik filtering.

In this paper we first consider stability properties of the higher order filtering methods. We mainly restrict ourselves to the one-dimensional case although some of the results can be carried over to higher dimensions. The main result presented here related to stability is that higher order nonlinear diffusion filtering is stable with respect to the L^2-norm. Furthermore we will see that there is an analogue to edge enhancement for fourth order nonlinear diffusion which will be called *curvature enhancement*. A closer look at various commonly used diffusivities shows that the behaviour for second and fourth order filtering is different but exhibits strong structural similarities.
This paper is organised as follows. Section 2 gives a short summary of useful properties of second order nonlinear diffusion related to this paper. In Section 3 we generalise the associated partial differential equation and its boundary conditions to higher orders and establish L^2-stability. The enhancement of local features is then discussed in Section 4 from a theoretical point of view. Section 5 displays some experiments confirming that fourth order forward and backward diffusion are practically observable effects. A summary of the main results concludes the paper with Section 6.

2 Second Order Nonlinear Diffusion

Let $f(x)$ denote a signal defined on an interval $[a,b]$. Second order nonlinear diffusion creates a filtered signal $u(x,t)$ as solution of the diffusion equation

$$\partial_t u = \partial_x \left(g \left((\partial_x u)^2 \right) \partial_x u \right) \tag{1}$$

with initial condition $u(x,0) = f(x)$ for all $x \in [a,b]$. The velocity of diffusion is steered by the diffusivity function $g \geq 0$ depending on the square of the first derivative of the evolving signal u. Typical diffusivities g tend to 1 for small absolute values of their argument and are getting smaller for higher argument modulus. This speeds up the diffusion in almost flat regions of the signal and reduces the diffusion speed near edges. With an appropriate choice for the diffusivity g, piecewise constant filtering results are possible. We will have a closer look at some typical diffusivities in Section 4.3. To complete the PDE (1) usually homogeneous Neumann boundary conditions $\partial_x u(a) = \partial_x u(b) = 0$ are assumed. There are two reasons for the appearance of this type of boundary conditions: Firstly they can be physically motivated by the idea that the flux is zero at the image boundaries. That means no matter is entering or leaving the image during the filtering process. Secondly Neumann boundary conditions emerge in a natural way if one relates nonlinear diffusion filtering to regularisation and energy functional minimisation [14]. This approach starts with an energy functional of the form

$$E_1(u) = \int_a^b \left((u-f)^2 + \alpha\, \varphi\left((\partial_x u)^2 \right) \right) dx$$

with a penaliser φ and weight $\alpha > 0$ and searches for a minimiser u. The first term $(u-f)^2$ is minimal if u is close to the initial image f in the sense of the L^2-norm. The second term $\varphi((\partial_x u)^2)$ with $\varphi(0) = 0$ and $\varphi' \geq 0$ rewards signals whose first derivative is small, i. e. it rewards smoothness of the filtered signal. A necessary condition for a minimiser of E_1 is given by the elliptic Euler-Lagrange equation

$$\frac{u-f}{\alpha} = \partial_x \left(\varphi'\left((\partial_x u)^2 \right) \partial_x u \right) \quad . \tag{2}$$

When no assumptions about the boundary behaviour are imposed, the derivation of the Euler-Lagrange equations leads to homogeneous Neumann boundary conditions. Interpreting the right-hand side of (2) as discretisation of a time derivative $\partial_t u$ and setting $g := \varphi'$, one ends up with the parabolic equation (1) with stopping time α. Later on we will see that this derivation helps us to find appropriate boundary conditions for the higher order diffusion filters, too. Stability in the L^2-norm and even a maximum-minimum principle belong to the properties which establish the good reputation of second order nonlinear diffusion methods [6].

3 Higher Order Nonlinear Diffusion

Higher order nonlinear diffusion filtering as considered in this paper is related to the equation

$$\partial_t u = (-1)^{m+1}\, \partial_x^m \left(g\left((\partial_x^m u)^2 \right) \partial_x^m u \right) \tag{3}$$

with initial condition $u(x,0) = f(x)$ for all $x \in [a,b]$. Since the highest derivative order appearing in (3) is $2m$, we will call the related process $2m$-th order nonlinear diffusion filtering. To determine appropriate boundary conditions we consider the derivation of (3) from the energy functional

$$E_m(u) = \int_a^b \left((u-f)^2 + \alpha\, \varphi\left((\partial_x^m u)^2 \right) \right) dx$$

following the train of thought of Section 2. A necessary condition for a minimiser u of this functional is given by the Euler-Lagrange equation

$$\frac{u-f}{\alpha} = (-1)^{m+1}\, \partial_x^m \left(g\left((\partial_x^m u)^2 \right) \partial_x^m u \right)$$

where we again set $g := \varphi'$. The corresponding natural boundary conditions are in this case

$$\partial_x^k \left(g\left((\partial_x^m u)^2 \right) \partial_x^m u \right) = 0 \quad \text{for } k \in \{0, \dots, m-1\} \tag{4}$$

for $x \in \{a, b\}$. We obtain m constraints at each boundary pixel as generalisation of the Neumann conditions for $m = 1$.
The subject of existence, uniqueness, and regularity of solutions will not be addressed in this paper. Usually a mollifier smoothing the argument of the diffusivity is required to obtain well-posedness [2,13]. Since the reasoning presented in this paper is independent of the presence of such a mollifier we omit it during the paper for simplicity. In the sequel we assume the existence, uniqueness and sufficient regularity of solutions. With these assumptions, the following proposition assures L^2-stability of the solutions:

Proposition 3.1. (L^2-Stability) *If a classical solution u of equation (3) exists which is continuously differentiable in the time variable t and $2m$ times continuously differentiable in the space variable $x \in [a, b]$, the L^2-norm of $u(\cdot, t)$ is monotonically decreasing with $t \geq 0$.*

Proof. Using the assumption that u satisfies (3) with the boundary conditions (4), integration by parts yields

$$\begin{aligned}
\partial_t \left(\frac{1}{2} \int_a^b u^2 \, dx \right) &= \int_a^b u \cdot (\partial_t u) \, dx \\
&= (-1)^{m+1} \int_a^b u \cdot \partial_x^m \left(g\left((\partial_x^m u)^2 \right) \partial_x^m u \right) dx \\
&= (-1)^{2m+1} \int_a^b (\partial_x^m u) \cdot g\left((\partial_x^m u)^2 \right) \cdot (\partial_x^m u) \, dx \\
&\quad + \sum_{k=0}^{m-1} (-1)^{m-1-k} \left[(\partial_x^k u) \cdot \partial_x^{m-1-k} \left(g\left((\partial_x^m u)^2 \right) \partial_x^m u \right) \right]_a^b \\
&= - \int_a^b g\left((\partial_x^m u)^2 \right) \cdot (\partial_x^m u)^2 \, dx \quad \leq \quad 0 \; .
\end{aligned}$$

This ensures that the L^2-norm of the solution may not increase with t. □

This result guarantees that higher order nonlinear diffusion leads indeed to a simplification of the initial data.

4 Local Feature Enhancement

Though classical nonlinear diffusion simplifies signals or images, it may also enhance important local features such as edges. This section discusses higher order diffusion from this point of view.

4.1 Second Order Filtering and Edge Enhancement

To determine the possibility of edge enhancement for special diffusivities g one usually uses the the flux function $\Phi(s^2) := g(s^2)s$ to rewrite (1) yielding

$$\partial_t u = \Phi'\left((\partial_x u)^2\right)\partial_x^2 u = \left(2g'\left((\partial_x u)^2\right)(\partial_x u)^2 + g\left((\partial_x u)^2\right)\right)\partial_x^2 u \ .$$

In regions where $\Phi'((\partial_x u)^2) > 0$ this equation behaves like a forward diffusion equation while in regions with $\Phi'((\partial_x u)^2) < 0$ there is backward diffusion possible. In this regions with backward diffusion, an edge enhancing behaviour is plausible and can also be observed in practice [1].

4.2 Fourth Order Filtering

Now we take a closer look at the fourth order diffusion equation, i. e. we set $m = 2$ in (3) yielding

$$\partial_t u = -\partial_x^2\left(g\left(\left(\partial_x^2 u\right)^2\right)\partial_x^2 u\right) \ .$$

We expand the right-hand side of this equation and rewrite it as

$$\partial_t u = -\left(2\left(\partial_x^3 u\right)^2 \Phi_1\left(\left(\partial_x^2 u\right)^2\right)\right)\partial_x^2 u - \Phi_2\left(\left(\partial_x^2 u\right)^2\right)\partial_x^4 u \qquad (5)$$

using $\Phi_1(s^2) := 2g''(s^2)s^2 + 3g'(s^2)$ and $\Phi_2(s^2) := 2g'(s^2)s^2 + g(s^2)$. Analogue to the second order case our argumentation is that (5) locally behaves similar to the linear equation $\partial_t u = -a\partial_x^2 u - b\partial_x^4 u$ if the signs of the factors a and b are equal to the signs of Φ_1 and Φ_2. For $\Phi_1((\partial_x^2 u)^2) < 0$ we expect some second order forward diffusion influence on the solution, whereas $\Phi_1((\partial_x^2 u)^2) > 0$ leads to second order backward diffusion. Vice versa, $\Phi_2((\partial_x^2 u)^2) > 0$ ensures fourth order forward diffusion, and $\Phi_2((\partial_x^2 u)^2) < 0$ fourth order backward diffusion.
It should be mentioned that Φ_2 always coincides with the function Φ in the second order case presented in Section 4.1. Also for orders higher than four, the sign of this function determines the diffusion property (forward or backward) of the highest order term which implies a certain similarity in the behaviour of several filtering orders. The main difference is the argument: Φ depends on the squared m-th derivative for $2m$-th order filtering.

4.3 Application to Commonly Used Diffusivities

After showing the general approach for fourth order diffusion in the last section we now apply it to several diffusivities commonly used in practice to describe their characteristic behaviour. In the following the diffusivities are ordered according to their forward-backward diffusion properties:

- **Forward Diffusion:** The diffusivity related to the regularisation approach by Charbonnier et al. [3] is given by $g(s^2) = \left(1 + \frac{s^2}{\lambda^2}\right)^{-\frac{1}{2}}$ and is known to perform forward diffusion in the second order case. By computing

$$\Phi_1(s^2) = -\tfrac{3}{2\lambda^2}\left(1 + \tfrac{s^2}{\lambda^2}\right)^{-\frac{5}{2}} < 0 \quad \text{and} \quad \Phi_2(s^2) = \left(1 + \tfrac{s^2}{\lambda^2}\right)^{-\frac{3}{2}} > 0$$

  we see that also the fourth order Charbonnier diffusion always performs forward diffusion. With the observation $\left(\varepsilon^2 + s^2\right)^{-\frac{1}{2}} = \varepsilon\left(1 + \frac{s^2}{\varepsilon^2}\right)^{-\frac{1}{2}}$ it is clear that regularised TV flow [15] behaves in the same way.
- **Boundary Case:** TV flow [5] comes from the diffusivity $g(s^2) = \frac{1}{|s|}$. At all points where the argument s is nonzero we have $\Phi_1(s^2) = \Phi_2(s^2) = 0$ which legitimates to consider TV flow as the boundary case between forward and backward diffusion.
- **Forward and Backward Diffusion:** The diffusivity $g(s^2) = \left(1 + \frac{s^2}{\lambda^2}\right)^{-1}$ proposed by Perona and Malik [1] leads to the conditions

$$\begin{aligned} \Phi_1(s^2) &= \tfrac{1}{\lambda^4}\left(1 + \tfrac{s^2}{\lambda^2}\right)^{-3}\left(s^2 - 3\lambda^2\right) < 0 \quad &\Longleftrightarrow \quad |s| < \sqrt{3}\lambda \\ \Phi_2(s^2) &= \left(1 + \tfrac{s^2}{\lambda^2}\right)^{-2}\left(1 - \tfrac{s^2}{\lambda^2}\right) \qquad > 0 \quad &\Longleftrightarrow \quad |s| < \lambda\ . \end{aligned}$$

  This really displays the adaptive nature of this diffusivity: Depending on the parameter λ, the curvature $\left|\partial_x^2 u\right|$ leads to forward or backward diffusion. New to the fourth order case is the presence of two conditions and the possibility that only one of them holds, namely in regions where $\lambda < \left|\partial_x^2 u\right| < \sqrt{3}\lambda$. Similar conditions hold for the diffusivity $g(s^2) = \exp\left(-\frac{s^2}{2\lambda^2}\right)$ also proposed by Perona and Malik [1].
- **Backward Diffusion:** The balanced forward-backward diffusivity [4] defined by $g(s^2) = \frac{1}{s^2}$ leads to $\Phi_1(s^2) = s^{-4} > 0$ and $\Phi_2(s^2) = -s^{-2} < 0$ which implies that it always performs backward diffusion. As for total variation diffusivity we also suppose that the argument is nonzero here.

We conclude that even in the fourth order case there are diffusivities covering the whole spectrum from pure forward to pure backward diffusion.

5 Numerical Examples

After the theoretical description of fourth order diffusion, in this section we show results of Perona-Malik filtering in one dimension with different orders. For $2m$-th order filtering with $g(s^2) = \left(1 + \frac{s^2}{\lambda^2}\right)^{-1}$ the parameter λ is chosen such that there are regions with $|\partial_x^m u| > \sqrt{3}\lambda$ where backward diffusion appears. Fig. 2 shows the initial signal and some filtering results. While second order filtering yields enhancement of edges, the fourth order filtering result tends to be piecewise linear with enhanced curvature at corner points. This observation for fourth order filtering is further affirmed by the almost piecewise constant derivative approximation of the filtering result also shown in Fig. 2.

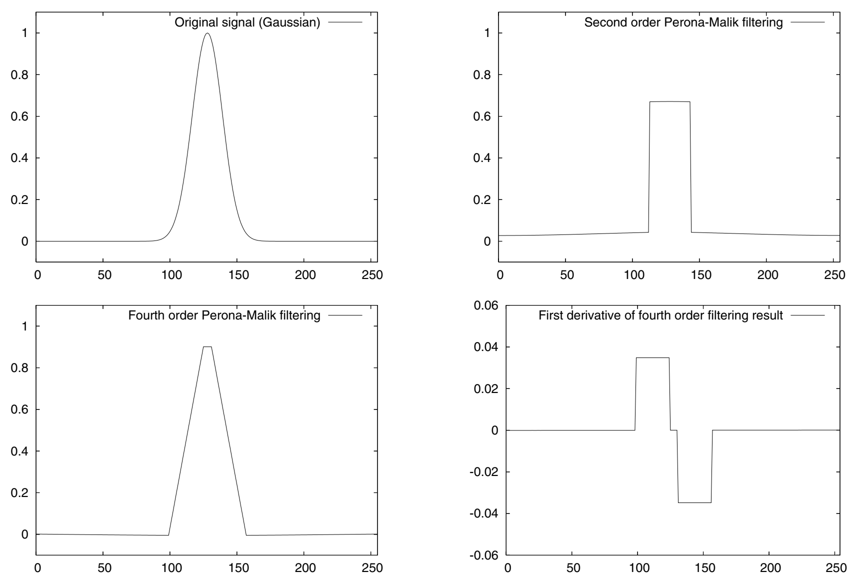

Fig. 2. *Top left:* Gaussian signal. *Top right:* Second order Perona-Malik filtering. *Bottom left:* Fourth order Perona-Malik filtering. *Bottom right:* First derivative of fourth order filtering result.

6 Conclusions

In this paper we have investigated theoretical properties of higher order nonlinear diffusion filters. For the first time we have presented stability considerations for a class of nonlinear diffusion filters related to variational methods. Furthermore, an argumentation in terms of forward and backward diffusion has been given which can be helpful to understand the behaviour of fourth order nonlinear diffusion filters. Numerical examples with one-dimensional data show that higher order filters can be used to enhance important data features. We have seen that in correspondence to the edge enhancement of second order diffusion, fourth order filters may act curvature enhancing, which is in accordance with the theoretical considerations presented in this paper.

A theoretical generalisation to orders higher than four and new practically usable diffusivities which are especially designed for higher orders are two questions of our ongoing research.

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References

1. Perona, P., Malik, J.: Scale space and edge detection using anisotropic diffusion. IEEE Transactions on Pattern Analysis and Machine Intelligence **12** (1990) 629–639
2. Catté, F., Loins, P.L., Morel, J.M., Coll, T.: Image selective smoothing and edge detection by nonlinear diffusion. SIAM Journal on Numerical Analysis **29** (1992) 182–193
3. Charbonnier, P., Blanc-Féraud, L., Aubert, G., Barlaud, M.: Deterministic edge-preserving regularization in computed imaging. IEEE Transactions on Image Processing **6** (1997) 298–311
4. Keeling, S.L., Stollberger, R.: Nonlinear anisotropic diffusion filtering for multiscale edge enhancement. Inverse Problems **18** (2002) 175–190
5. Andreu, F., Ballester, C., Caselles, V., Mazón, J.M.: Minimizing total variation flow. Differential and Integral Equations **14** (2001) 321–360
6. Weickert, J.: Anisotropic Diffusion in Image Processing. B. G. Teubner, Stuttgart (1998)
7. Aubert, G., Kornprobst, P.: Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations. Volume 147 of Applied Mathematical Sciences. Springer, New York (2002)
8. You, Y.L., Kaveh, M.: Fourth-order partial differential equations for noise removal. IEEE Transactions on Image Processing **9** (2000) 1723–1730
9. Chan, T., Marquina, A., Mulet, P.: High-order total variation-based image restoration. SIAM Journal of Scientific Computing **22** (2000) 503–516
10. Lysaker, M., Lundervold, A., Tai, X.C.: Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time. IEEE Transactions on Image Processing **12** (2003) 1579–1590
11. Scherzer, O.: Denoising with higher order derivatives of bounded variation and an application to parameter estimation. Computing **60** (1998) 1–27
12. Tumblin, J., Turk, G.: LCIS: A boundary hierarchy for detail-preserving contrast reduction. In: SIGGRAPH '99: Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, ACM Press/Addison-Wesley (1999) 83–90
13. Greer, J.B., Bertozzi, A.L.: H^1 solutions of a class of fourth order nonlinear equations for image processing. Discrete and Continuous Dynamical Systems **10** (2004) 349–368
14. Scherzer, O., Weickert, J.: Relations between regularization and diffusion filtering. Journal of Mathematical Imaging and Vision **12** (2000) 43–63
15. Feng, X., Prohl, A.: Analysis of total variation flow and its finite element approximations. ESAIM: Mathematical Modelling and Numerical Analysis **37** (2003) 533–556