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A General Structure Tensor Concept and Coherence-Enhancing Diffusion Filtering for Matrix Fields

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Abstract

Coherence-enhancing diffusion filtering is a striking application of the structure tensor concept in image processing. The technique deals with the problem of completion of interrupted lines and enhancement of flow-like features in images. The completion of line-like structures is also a major concern in diffusion tensor magnetic resonance imaging (DT-MRI). This medical image acquisition technique outputs a 3D matrix field of symmetric 3×3-matrices, and it helps to visualise, for example, the nerve fibers in brain tissue. As any physical measurement DT-MRI is subjected to errors causing faulty representations of the tissue corrupted by noise and with visually interrupted lines or fibers.

In this paper we address that problem by proposing a coherence-enhancing diffusion filtering methodology for matrix fields. The approach is based on a generic structure tensor concept for matrix fields that relies on the operator-algebraic properties of symmetric matrices, rather than their channel-wise treatment of earlier proposals.

Numerical experiments with artificial and real DT-MRI data confirm the gap-closing and flow-enhancing qualities of the technique presented.

Keywords: matrix field, symmetric matrix, diffusion tensor MRI, coherence-enhancing diffusion filtering, CED, structure tensor

1 Introduction

One of the most pressing tasks in diffusion tensor magnetic resonance imaging (DT-MRI) is the so-called fiber tracking [21]. In parts of the brain, e.g. the corpus callosum, nerve fibers form bundles with a coherent structure. In principle the matrix field produced by DT-MRI allows for the application of fiber tracking techniques and hence for the accurate visualisation of the nerve fibers. There is a vast literature on various techniques to achieve this goal, see for example [10, 23, 8, 3, 22, 14, 11]. However, ubiquitous measurement errors during acquisition cause gaps and interrupted fibers in the final visualisation. It would be desirable to have a method available that enhances flow-like patterns such as bundles of nerve fibers.

For scalar and vector-valued images a method achieving this goal is already at our disposal: Coherence enhancing diffusion filtering [19].

It is governed by the equation

$$\begin{aligned}
\partial_t u - \operatorname{div}\left(D \cdot \nabla u\right) &= 0 && \text{in } I \times \Omega, \\
\partial_n u &= 0 && \text{in } I \times \partial\Omega, \\
u(x,0) &= f(x) && \text{in } \Omega,
\end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is the image domain and $I = [0, T[$ a potentially unbounded time interval.

To the authors' best knowledge, no extension of this method to matrix-valued images, matrix fields for short, has been reported in the literature.

The essential ingredient in this equation is the *diffusion tensor D* of the scalar image u which steers the diffusion process: It amplifies diffusion along flow-like structures, and hinders diffusion perpendicular to those pattern. Postponing the detailed construction of D to the subsequent Section 2, for now we only remark that it is a function of the structure tensor [12] which is given by

$$S_\rho(u(x)) := G_\rho * \left(\nabla u(x) \cdot (\nabla u(x))^\top\right) = \left(G_\rho * \left(\partial_{x_i} u(x) \cdot \partial_{x_j} u(x)\right)\right)_{i,j=1,\ldots,d}$$

Here $G_\rho *$ indicates a convolution with a Gaussian of standard deviation ρ, however, more general averaging procedures can be used. If $\nabla u(x) \neq 0$ the matrix

$$\left(\nabla u(x) \cdot (\nabla u(x))^\top\right)$$

has rank one, the eigenvector $\nabla u(x)$ belongs to the only non-zero eigenvalue $|\nabla u(x)|^2$. The eigenvalues represent the contrast in the directions of the eigenspaces. The averaging process then creates a matrix with full rank which contains valuable directional information. Note that the averaging of the structure tensor avoids cancellation of directional information. If one would average the gradients instead, neutralisation of vectors with opposite sign would occur. In many applications it is advantageous to use a presmoothed image $u_\sigma := G_\sigma * u$ instead of u in order to reduce the influence of noise for better numerical results. The structure tensor is a classical tool in image processing to extract directional information from an image, for more details the reader is referred to [2] and the literature cited there.

It is not straightforward to generalise both the structure and the diffusion tensor concept to the setting of matrix-valued images.

To fix notation in this work matrix-valued images or matrix fields $M(x)$ are considered as mappings from $\mathbb{R}^d$ into the set $\operatorname{Sym}_n(\mathbb{R})$ of symmetric $n \times n$-matrices

$$M : \; x \mapsto M = \left(m_{i,j}(x)\right)_{i,j=1,\ldots,n} \in \operatorname{Sym}_n(\mathbb{R}).$$

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and denoted by capital letters while indexed lower case letters indicate their components.

In [20, 4] di Zenso's approach to a structure tensor for multi channel images is generalised: Each channel considered as independent scalar image gives rise to a structure tensor, then these structure tensors are summed up to give the structure tensor of Weickert

$$J_\rho(U(x)) := \sum_{i,j=1}^{n} S_\rho(u_{i,j}(x))$$

This construction has been refined to a customisable structure tensor in [15]. There the resulting structure tensor is a weighted sum of tensors of scalar quantities that are now not just the channels, but other meaningful scalar quantities derived from the matrix field. The weights are provided by the user, and depending on the choice of weights the emerging structure tensor has a sensitivity for certain features of the matrix field. A special constellation of the weights turns the customisable structure tensor into Weickert's structure tensor. It is important to mention that in case of a 3D matrix field of 3×3 symmetric matrices these concepts yield also 3×3 structure tensor, the very same order as a 3D scalar image.

Here we opt for a different approach: We assume an operator-algebraic view on symmetric matrices as finite dimensional instances of selfadjoint Hilbert space operators. The exploitation of the algebraic properties of matrices, ensures proper interaction between the different matrix channels. This is a decisive advantage over the standard componentwise treatment of vector-valued images.

Promising proposals to generalise nonlinear regularisation methods and related diffusion filters for scalar images to matrix fields have been made in [7, 6]. These approaches are based on a basic differential calculus for matrix fields, which will be useful in this context as well. Other approaches to tensor field regularisation have a more differential geometric background [16, 9] where the set of positive definite matrices is endowed with a Riemannian metric stemming form the DT-MRI field.

In this paper we will present a general concept for a large size structure tensor that carries all the directional information of the matrix field. We will show how this information can be deduced from this large tensor by a reduction process. This reduction process illuminates also its connection to the structure tensors mentioned above.

The article is structured as follows: The next Section 2 is devoted to a brief review of coherence enhancing diffusion (CED) filtering of scalar images. Notions necessary to construct the diffusion tensor and A basic differential

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calculus for matrix fields necessary to construct the diffusion tensor is provided in Section 3. In Section 4 we propose a novel structure tensor concept for matrix fields, study some of its properties by investigating the connection to already known structure tensors for matrix-valued data. We then feature the potential of this concept by proposing a coherence enhancing diffusion for matrix fields in Section 5. We report on the results of our experiments with matrix-valued coherence enhancing diffusion applied to real DT-MRI images in Section 6. Section 7 is made up by concluding remarks.

2 Synopsis of Coherence Enhancing Diffusion

The rationale behind the construction of the diffusion tensor D is as follows: The matrix $S_\rho(u)$ as the positive average of different symmetric positive semidefinite matrices has the very same property. Hence $S_\rho(u)$ has a orthonormal system $\{w_1, \ldots, w_d\}$ of eigenvectors corresponding to the non-negative eigenvalues $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_d \geq 0$ indicating the contrast in each direction. In the line defined by w_d, the *coherence orientation* the contrast is the least compared to other orientations, since w_d belongs to the smallest eigenvalue μ_d. The coherence or anisotropy of an image structure essentially is captured in eigenvalue distribution of the structure tensor S_ρ. In [18] the quantity

$$\kappa := \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} (\mu_i - \mu_j)^2$$

is proposed to measure coherence. Strongly differing eigenvalues result in a large value of κ, while similar values produce a small κ-value indicating a structure with isotropic character. The matrix D has the same eigenvectors as S_ρ, however, its eigenvectors λ_i are altered via the tensor transfer map H according to

$$\lambda_i := H(\mu_i) := \alpha \qquad \text{for } i = 1, \ldots, d-1$$

and

$$\lambda_d := H(\mu_i) := \begin{cases} \alpha & \text{if } \kappa = 0, \\ \alpha + (1-\alpha) \exp\left(-\frac{C}{\kappa}\right) & \text{else.} \end{cases}$$

with a threshold $C > 0$. Note that $\min\{\lambda_i, i = 1..d\} = \alpha > 0$, which causes D to be uniformly positive definite and enforces a diffusion no matter how isotropic ($\kappa \downarrow 0$) the image structure becomes.

It is our task in this paper to extend the notion of diffusion tensor from scalar images to matrix fields. Ultimately we aim at coherence enhancing diffusion filtering of matrix fields. To do so we have to clarify what is meant by the

partial derivative, the gradient and, most of all, the suitable structure tensor of a matrix field. This is done in the next section.

3 Basic Differential Calculus for Matrix Fields

In this section we provide briefly the *basic definitions* for the formulation of a differential calculus for matrix fields. This material is instigated in [5] but for a more detailed exposition the reader is referred to [7].

1. **Functions of matrices.** The standard definition of a function h on $\text{Sym}_n(\mathbb{R})$ is given by [13]:

$$h(U) = V^\top \text{diag}(h(\lambda_1), \ldots, h(\lambda_n))V \in \text{Sym}_n(\mathbb{R}),$$

if $U = V^\top \text{diag}(\lambda_1, \ldots, \lambda_n)V$ is the spectral/eigen decomposition of the symmetric matrix U, and if $\lambda_1, \ldots, \lambda_n$ lie in the domain of definition of h. We encountered already an example of a function of a symmetric matrix; the diffusion tensor as a function of the structure tensor S_ρ with coherence κ under the tensor transfer map H, $D = H(S_\rho)$.

2. **Partial derivatives.** Let $\omega \in \{x_1, \ldots, x_d, t\}$ stand for a spatial or temporal variable, and set $(x, t) = (x_1, \ldots, x_d, t)$. h partial derivative for matrix fields is naturally defined *componentwise* as the limit of a difference quotient:

$$
\begin{aligned}
\overline{\partial}_\omega U(x,t) &= \lim_{h \to 0} \frac{U((x,t) + h \cdot e_k) - U(x,t)}{h} \\
&= \left(\lim_{h \to 0} \frac{u_{ij}((x,t) + h \cdot e_k) - u_{ij}(x,t)}{h} \right)_{i,j} \\
&= \left(\partial_\omega u_{ij}(x,t) \right)_{i,j}
\end{aligned}
$$

where $e_k := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ stands for the kth unit vector of space-time $\mathbb{R}^{d+1}$. The generalisation to directional derivatives is straight forward, then ω would denote an appropriate unit vector. Higher order partial differential operators, such as the Laplacian, or other more sophisticated operators, find their natural counterparts in the matrix-valued framework in this way as well. It is worth mentioning that for the operators $\overline{\partial}_\omega$ a *product rule* holds:

$$\overline{\partial}_\omega(A(x) \cdot B(x)) = (\overline{\partial}_\omega A(x)) \cdot B(x)) + A(x) \cdot (\overline{\partial}_\omega B(x)).$$

3. **Generalized gradient of a matrix field.** The gradient of a matrix field with sufficiently smooth component functions is defined via

$$\overline{\nabla}U(x) := (\overline{\partial}_{x_1} U(x), \dots, \overline{\partial}_{x_d} U(x))^\top \in (\mathrm{Sym}_n(\mathbb{R}))^d$$

Hence, the generalised gradient $\overline{\nabla}U(x)$ at a voxel x is regarded as an element of the module $(\mathrm{Sym}_n(\mathbb{R}))^d$ over $\mathrm{Sym}_n(\mathbb{R})$ in close analogy to the scalar setting where $\nabla u(x) \in \mathbb{R}^d$. In the sequel we will call a mapping from R^d into $(\mathrm{Sym}_n(\mathbb{R}))^d$ a *module field* rather than a vector field. Note that this definition of a generalised gradient is different from one that might be expected when viewing a matrix as a tensor (of second order). According to differential geometry concepts their derivatives are tensors of third order. However, we adopt a operator-algebraic point of view: The matrices are self-adjoint operators that can be added, multiplied with a scalar, and concatenated. Thus, they form an algebra, and we aim at consequently replacing the field $\mathbb{R}$ by the algebra $\mathrm{Sym}_n(\mathbb{R})$ in the scalar, that is, $\mathbb{R}$-based formulation of differential calculus.

4. For the sake of completeness we include the formal definition of the **generalized structure tensor of a matrix field** here. We will discuss its derivation, properties and application in the next section. The novel structure tensor for a matrix field is given by

$$\begin{aligned}
\overline{\mathcal{S}}_L\left(U(x)\right) &:= G_\rho * \left(\overline{\nabla}U(x) \cdot (\overline{\nabla}U(x))^\top\right) \\
&= \left(G_\rho * \left(\overline{\partial}_{x_i}U(x) \cdot \overline{\partial}_{x_j}U(x)\right)\right)_{i,j=1,\dots,d}.
\end{aligned} \tag{2}$$

5. **Symmetrised product of symmetric matrices.** The product of two symmetric matrices $A, B \in \mathrm{Sym}_n(\mathbb{R})$ is not symmetric unless the matrices commute. However, it is vital to our interests to have a symmetrised matrix product at our disposal. There are numerous options to define a symmetrised matrix product, however, we concentrate on a specific one known from algebra and called Jordan product:

$$A \bullet_J B = \frac{1}{2}(AB + BA) \quad \text{for } A, B \in \mathrm{Sym}_n(\mathbb{R}). \tag{3}$$

For commuting A and B we have $A \bullet_J B = A \cdot B$. This product is commutative and distributive but not associative. It is one half of the anti-commutator of A and B, but due to its additive structure no determinant product rule holds. Most important, it does not preserve the positive semidefinitness of its arguments as the following simple example shows:

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \bullet_J \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \left(\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} \right) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1$

Remarks:

1) It should be mentioned that the logarithmic multiplication introduced in [1] and given by $A \bullet_L B := \exp(\log(A) + \log(B))$ is defined only for positive definite matrices. However, the matrix-valued diffusion proposed here requires the multiplication to be able to cope with the factor matrices being indefinite. Furthermore matrix fields that are not necessarily positive semidefinite should also be within the reach of our PDE-based filtering. Hence the logarithmic multiplication is not suitable for our purpose.

2) The proposed notions for a calculus on symmetric matrix fields are extensions of the calculus of scalar multivariate functions. As such it must be possible to regain the scalar calculus from the newly introduced matrix-valued framework by specification. There are two ways to view scalar calculus as a special case of the matrix calculus: Clearly, setting $n = 1$ turns the matrix field into a scalar function. However, one can also embed the set of real numbers $\mathbb{R}$ into the set of symmetric matrices $\mathrm{Sym}_n(\mathbb{R})$ by the identification $\mathbb{R} \ni r \longleftrightarrow r \cdot I_n$ with the $n \times n$ identity matrix I_n. Hence, aside from having a certain simplicity, it is mandatory that the proposed extensions collapse to the scalar calculus when making the transition from scalar functions to matrix fields in one way or the other.

We summarised the definitions from above and juxtapose them with their scalar counterparts in the subsequent small table 3. The matrix field $U(x)$ is assumed to be diagonisable with $U = (u_{ij})_{ij} = V^\top \mathrm{diag}(\lambda_1, \ldots, \lambda_n)V$, where $V \in O(n)$, the set of all orthogonal $n \times n$-matrices, and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Setting	scalar valued	matrix-valued
function	$h : \begin{cases} \mathbb{R} \longrightarrow \mathbb{R} \\ x \mapsto h(x) \end{cases}$	$h : \begin{cases} \mathrm{Sym}_n(\mathbb{R}) \longrightarrow \mathrm{Sym}_n(\mathbb{R}) \\ U \mapsto V^\top \mathrm{diag}(h(\lambda_1), \ldots, h(\lambda_n))V \end{cases}$
partial derivatives	$\partial_\omega u,$ $\omega \in \{t, x_1, \ldots, x_d\}$	$\overline{\partial}_\omega U := \left(\partial_\omega u_{ij}\right)_{ij},$ $\omega \in \{t, x_1, \ldots, x_d\}$
gradient	$\nabla u(x) := (\partial_{x_1} u(x), \ldots, \partial_{x_d} u(x))^\top,$ $\nabla u(x) \in \mathbb{R}^d$	$\overline{\nabla}U(x) := (\overline{\partial}_{x_1} U(x), \ldots, \overline{\partial}_{x_d} U(x))^\top,$ $\overline{\nabla}U(x) \in (\mathrm{Sym}_n(\mathbb{R}))^d$
structure tensor	$G_\rho * \left(\overline{\nabla}u(x) \cdot (\overline{\nabla}u(x))^\top\right)$	$\overline{\mathcal{S}}_L\left(U(x)\right) := G_\rho * \left(\overline{\nabla}U(x) \cdot (\overline{\nabla}U(x))^\top\right)$
product	$a \cdot b$	$A \bullet_J B := \frac{1}{2}(AB + BA)$

Table 1: Extensions of elements of scalar valued calculus **(middle)** to the matrix-valued setting **(right)**.

4 The Structure Tensor $\overline{\mathcal{S}}_L$ for Matrix Fields

4.1 Derivation of $\overline{\mathcal{S}}_L$

With the terminology introduced above we infer for the directional derivative of $x \mapsto U(x)$ in direction $v \in S^{d-1}$:

$$
\begin{aligned}
\overline{\partial}_v U(x) & := & \frac{d}{dh} U(x + h \cdot v) \big|_{h=0} = \sum_{k=1}^{d} (\partial_{x_k} u_{ij}(x))\, v_k \\
& = & (\overline{\nabla}U(x))^\top \begin{pmatrix} v_1\, I_n \\ \vdots \\ v_d\, I_n \end{pmatrix} \\
& = & v_d\, \overline{\partial}_{x_1} U(x) + \cdots + v_d\, \overline{\partial}_{x_d} U(x) \in \mathrm{Sym}_n(\mathbb{R})
\end{aligned}
$$

This expression is a symmetric matrix but aside from that in complete analogy to the corresponding real-valued term in the scalar case. In the scalar setting the direction of steepest descent/ascent would be be given by the direction that optimises the directional derivative. However, in the matrix valued setting the entries of the generalised gradient are matrices and finding an optimal unit vector v that optimises the matrix-valued directional derivative is hindered by practical as well as theoretical obstacles: An optimisation relies on the presence of a total ordering, but on $\mathrm{Sym}_n(\mathbb{R})$ only partial ordering relations do exist. And even after choosing an partial ordering,e.g. the Loewner ordering, it is not clear how to obtain the optimal v in a reasonable

computational effort. A way out of this difficulty is the projection of the set of symmetric matrices $\text{Sym}_n(\mathbb{R})$ onto the real numbers by a so-called linear form. Each linear form on $\text{Sym}_n(\mathbb{R})$ is of the form of a scalar product,

$$M \longmapsto \langle A, M \rangle := \text{tr}_A(M) := \text{tr}(A \cdot M)$$

with a matrix $A \in \text{Sym}_n(\mathbb{R})$, [13]. Then one simply has

$$\text{argmax}\{\text{tr}_A(\overline{\partial}_v U(x)) \mid \|v\| = 1\} =$$
$$\frac{1}{\sqrt{\sum_1^d (\text{tr}_A(\overline{\partial}_{x_i} U(x)))^2}} \left(\text{tr}_A(\overline{\partial}_{x_1} U(x)), \ldots, \text{tr}_A(\overline{\partial}_{x_d} U(x)) \right).$$

We write $\text{tr} = \text{tr}_I$. Depending on the choice of A we obtain the direction of strongest change at one point x in the matrix field, or which boils down to the same thing: The strongest change in the scalar image $x \longmapsto \text{tr}_A(U(x))$. Apparently this approach suffers from the same weakness as the direct direction estimation in scalar images, the danger of cancellation through averaging. This reveals the need for a structure tensor for matrix fields and also its basic construction principle. It is close at hand to define a structure tensor for matrix fields as follows:

$$\overline{\mathcal{S}}_L(U(x)) := G_\rho * \left(\overline{\nabla} U(x) \cdot (\overline{\nabla} U(x))^\top \right) = \left(G_\rho * \left(\overline{\partial}_{x_i} U(x) \cdot \overline{\partial}_{x_j} U(x) \right) \right)_{i,j=1,\ldots,d}$$

Here $G_\rho *$ indicate a convolution with a Gaussian of standard deviation ρ or, more general, another appropriate averaging procedure. The parameter ρ is suppressed to avoid notational clutter. We list some immediate properties of this construct:

1. $\overline{\mathcal{S}}_L(U(x))$ is a symmetric $nd \times nd$-block matrix with d^2 blocks of size $n \times n$, $\overline{\mathcal{S}}_L(U(x)) \in \text{Sym}_d(\text{Sym}_n(\mathbb{R})) = \text{Sym}_{nd}(\mathbb{R})$. The symmetry follows from the fact that for all $i, j = 1, \ldots, d$

$$\begin{aligned}
(\overline{\partial}_{x_i} U(x) \cdot \overline{\partial}_{x_j} U(x))^\top &= (\overline{\partial}_{x_j} U(x))^\top \cdot (\overline{\partial}_{x_i} U(x))^\top \\
&= \overline{\partial}_{x_j} U(x) \cdot \overline{\partial}_{x_i} U(x).
\end{aligned}$$

2. The structure tensor $\overline{\mathcal{S}}_L$ can be diagonalised as

$$\overline{\mathcal{S}}_L(U) = \sum_{k=1}^{nd} \rho_k w_k w_k^\top$$

with real eigenvalues λ_k (w.l.o.g. arranged in decreasing order) and an orthonormal basis $\{v_k\}_{i=1,\ldots,nd}$ of $\mathbb{R}^{nd}$.

The usefulness of this construction will rise and fall with an positive answer to the following question:

How can we extract useful d-dimensional directional information from this large structure tensor $\overline{\mathcal{S}}_L$?

In the case of of a structure tensor $S \in \mathrm{Sym}_d^+(\mathbb{R})$ for a scalar image its eigenvectors provide all the important directional information of the image. However, in the matrix-valued setting the eigenvectors v_i of $\overline{\mathcal{S}}_L(U)$ are nd-dimensional vectors and they lack immediate physical interpretation. Hence, as such they do not provide really useful d-dimensional directional information. Instead, let us find an analog to the Rayleigh quotient of S

$$\mathrm{argmax}\ \{v^\top \left(G * \nabla u\ \nabla u^\top\right) v \mid v \in \mathbb{R}^d,\ \|v\| = 1\}$$

in the matrix field framework. We consider

$$(v_1\,I_n \cdots v_d\,I_n)\overline{\mathcal{S}}_L\left(U(x)\right) \begin{pmatrix} v_1\,I_n \\ \vdots \\ v_d\,I_n \end{pmatrix}$$

$$= \begin{pmatrix} v_1^2\,G_\rho * (\overline{\partial}_{x_1} U)^2 & \cdots & v_1 v_2\,G_\rho * (\overline{\partial}_{x_1} U \cdot \overline{\partial}_{x_d} U) \\ \vdots & \ddots & \vdots \\ v_1 v_d\,G_\rho * (\overline{\partial}_{x_d} U \cdot \overline{\partial}_{x_1} U) & \cdots & v_d^2\,G_\rho * (\overline{\partial}_{x_d} U)^2 \end{pmatrix}$$

Now we have to choose the real numbers $v_1, \ldots, v_d$ in an *optimal* way. Again we are facing difficulties caused by the absence of total ordering for matrices. In the case of a scalar image u (a matrix field with 1×1-matrices) the optimal v is the unit eigenvector corresponding to the largest eigenvalue of the structure tensor S. It will turn out to be convenient to utilise the notion of operator matrix.

The idea is to *reduce* $\overline{\mathcal{S}}_L(U) \in \mathrm{Sym}_{nd}(\mathbb{R})$ to a structure tensor $S(U) \in \mathrm{Sym}_n(\mathbb{R})$ in a generalised projection step employing the block operator matrix

$$\mathrm{Tr}_A := \begin{pmatrix} \mathrm{tr}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{tr}_A \end{pmatrix} \tag{4}$$

containing the trace operation. Again we set $\mathrm{Tr} := \mathrm{Tr}_I$. This operator matrix acts on elements of the space $(\mathrm{Sym}_n(\mathbb{R}))^d$ as well as on block matrices via formal blockwise matrix multiplication.

$$\begin{pmatrix} \mathrm{tr}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{tr}_A \end{pmatrix} \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{n1} & \cdots & M_{nn} \end{pmatrix} = \begin{pmatrix} \mathrm{tr}_A(M_{11}) & \cdots & \mathrm{tr}_A(M_{1n}) \\ \vdots & \ddots & \vdots \\ \mathrm{tr}_A(M_{n1}) & \cdots & \mathrm{tr}_A(M_{nn}) \end{pmatrix},$$

provided that the square blocks M_{ij} are compatible with tr_A, that means here, have the same size as A. The reason for choosing tr_A as reduction operators is their homogeneity:

$$\mathrm{tr}_A(t\ M) = t\ \mathrm{tr}_A(M) \quad \text{for all } t \in \mathbb{R}.$$

Note that $A = I$ provides the leading example since

$$\begin{pmatrix} \mathrm{tr}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{tr}_A \end{pmatrix} \begin{pmatrix} M_{11} & \cdots & M_{1d} \\ \vdots & \ddots & \vdots \\ M_{d1} & \cdots & M_{dd} \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{tr} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{tr} \end{pmatrix} \begin{pmatrix} A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A \end{pmatrix} \begin{pmatrix} M_{11} & \cdots & M_{1d} \\ \vdots & \ddots & \vdots \\ M_{d1} & \cdots & M_{dd} \end{pmatrix}$$

The subsequent result gives a first insight into the role of this reduction operation and its connection to other structure tensors:

Proposition: (Weickert's Tensor as an Elementary Reduction of $\overline{\mathcal{S}}_L$)
Let $U(x) \in \mathrm{Sym}_n(\mathbb{R})$ be a d-dimensional matrix-field. Then the Weickert tensor J_ρ is a reduced version of $\overline{\mathcal{S}}_L$,

$$\mathrm{Tr}\ \overline{S_L}(U) = J_\rho(U) \in \mathrm{Sym}_n(\mathbb{R}).$$

Proof: First we realise that we can disregard the convolution with a Gaussian G_ρ with integration scale ρ or any other linear averaging process since the trace operation commutes with such linear mappings. Therefor we can deal with the partial derivatives $\partial_{x_i} u_{i,j}$ of the matrix components directly. Fix $p, q \in \{1, \ldots, d\}$. Then the (p, q)th component of Weickert's structure tensor is given by

$$J_\rho(U) = \sum_{i,j}^{d} \partial_{x_p} u_{i,j} \cdot \partial_{x_q} u_{i,j}.$$

However, we obtain for the (p, q)th component of the *reduced version* of the large structure tensor $S_L(U)$

$$\mathrm{tr}(\overline{\partial}_{x_p} U \overline{\partial}_{x_q} U) = \sum_{i=1}^{d} \left(\overline{\partial}_{x_p} U \overline{\partial}_{x_q} U\right)_{i,i} = \sum_{i=1}^{d}\sum_{j=1}^{d}(\partial_{x_p} u_{i,j} \cdot \partial_{x_q} u_{j,i})$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d}(\partial_{x_p} u_{i,j} \cdot \partial_{x_q} u_{j,i}),$$

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where the symmetry of the matrix U accounts for the last equality. This proves the assertion.

The reduction operation is accompanied by an extension operation defined via the *Kronecker product*:

Definition: (Extension via Kronecker product)
The I_n-extension operation is the mapping from $\mathrm{Sym}_d(\mathbb{R})$ to $\mathrm{Sym}_{nd}(\mathbb{R})$ given by the Kronecker product $\otimes$:

$$
\begin{pmatrix} v_{11} & \cdots & v_{1d} \\ \vdots & \ddots & \vdots \\ v_{d1} & \cdot & v_{dd} \end{pmatrix} \longmapsto \begin{pmatrix} v_{11} & \cdots & v_{1d} \\ \vdots & \ddots & \vdots \\ v_{d1} & \cdots & v_{dd} \end{pmatrix} \otimes \begin{pmatrix} I_n & \cdots & I_n \\ \vdots & \ddots & \vdots \\ I_n & \cdots & I_n \end{pmatrix}
$$

$$
:= \begin{pmatrix} v_{11}I_n & \cdots & v_{1d}I_n \\ \vdots & \ddots & \vdots \\ v_{d1}I_n & \cdots & v_{dd}I_n \end{pmatrix}.
$$

If the $d \times d$-matrix $(v_{ij})_{ij}$ is Kronecker-multiplied with

$$
\begin{pmatrix} C & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C \end{pmatrix} \begin{pmatrix} I_n & \cdots & I_n \\ \vdots & \ddots & \vdots \\ I_n & \cdots & I_n \end{pmatrix} = \begin{pmatrix} C & \cdots & C \\ \vdots & \ddots & \vdots \\ C & \cdots & C \end{pmatrix}
$$

we speak of a C-extension.

4.2 A novel diffusion tensor $\overline{D}$ for matrix fields

Now it is possible to give an analog $\overline{D}$ to the diffusion tensor D in the framework of matrix fields. We proceed in four steps:

1. The matrix field $\mathbb{R}^d \ni x \mapsto U(x)$ provides us with an module field of generalised gradients $\overline{\nabla}U(x)$ from which we construct the generalised structure tensor $\overline{\mathcal{S}}_L U(x)$ possibly with a certain integration scale ρ. This step corresponds exactly to the scalar case.

2. We infer reliable d-dimensional directional information by reducing $\overline{\mathcal{S}}_L U(x)$ with tr_A with the help of the block operator matrix given in (4) leading to a symmetric $d \times d$-matrix S, for example $S = J_\rho$ if $A = I_n$.

$$
S := \begin{pmatrix} \mathrm{tr}_A & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{tr}_A \end{pmatrix} \overline{\mathcal{S}}_L U(x).
$$

3. The symmetric $d \times d$-matrix S is spectrally decomposed, and the tensor transfer map H is applied to S yielding the diffusion tensor D,

$$D := H(S).$$

4. Finally we enlarge the $d \times d$-matrix D to a $nd \times nd$-matrix $\overline{D}$ by the extension operation:

$$\overline{D} = D \otimes \left(\begin{pmatrix} C & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C \end{pmatrix} \begin{pmatrix} I_n & \cdots & I_n \\ \vdots & \ddots & \vdots \\ I_n & \cdots & I_n \end{pmatrix} \right).$$

This last step gives another possibility to steer the filter process by the choice of the matrix C. However, this is the subject of current research. For this work we restricted ourselves to $C = I_n$.

5 Coherence-Enhancing Diffusion Filtering for Matrix Fields

Now we have gathered the necessary ingredients to formulate the matrix-valued equivalent to the scalar coherence enhancing diffusion as expressed in equation (1).

$$\overline{\partial}_t U - \sum_{i=1}^{d} \overline{\partial}_{x_i} \left(\overline{D} \bullet \overline{\nabla} U \right) = 0 \qquad \text{in } I \times \Omega,$$

$$U(x, 0) = F(x) \qquad \text{in } \Omega,$$

Note that the Jordan-multiplication in $\overline{D} \cdot \overline{\nabla} U$ is understood in the block-wise sense of partitioned matrices. Moreover, we translated the divergence differential operator acting on a vector-valued function $u = (u_1, \ldots, u_d)$ div $u = \sum_{i=1}^{d} \partial_{x_i} u$ into its matrix-valued counterpart acting on a module field $W \in \text{Sym}_n(\mathbb{R})^d$ by

$$\overline{\text{div}} \ W = \sum_{i=1}^{d} \overline{\partial}_{x_i} W.$$

The numerical algorithm is inspired by the explicit scheme in [17]. We used a matrix-valued version employing the calculus framework for matrix fields as presented before.

6 Experiments

We use two data sets in our numerical experiments: The artificial matrix fields of 3×3-symmetric matrices exhibit various coherent structures ranging from simple line-like to curved features, figures 1–4. Important is the fact that these structures are not complete but interrupted. We will use these data to demonstrate the gap-closing and enhancing properties of our technique. The other matrix fields stems from a 2-D slice extracted from a 3-D DT-MRI data set of size of a $128 \times 128 \times 30$ of a human head, see Fig. 5.

The data are represented as ellipsoids via the level sets of the quadratic form $\{x^\top A^{-2} x = const. : x \in \mathbb{R}^3\}$ associated with a matrix $A \in \mathrm{Sym}^+(3)$. By using A^{-2} the length of the semi-axes of the ellipsoid correspond directly with the three eigenvalues of the matrix. We have added random positive definite matrices to the data to demonstrate the denoising capabilities of our coherence enhancing filtering, CED-filtering for short. The eigenvectors of this noise were obtained by choosing Gaussian-distributed numbers with standard deviation $\sigma = 1000.0$ and taking the absolute value for positive definiteness. The high standard deviation can be explained by the fact that in real-world data the typical eigenvalues are in the order of magnitude of 1000. The eigenvectors of the artificial noise result in choosing three uniformly distributed angles and rotating the matrix by these angles around the coordinate axes. The resulting data is shown in Fig. 2.

The artificial data set displayed in Fig. 1 imitates a crossing of nerve fibers. Depending on the choice of the reduction matrix A in tr_A either the diagonal directed downward, $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, or the one directed upward, $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, is given preference in the CED-filtering results. If no priority is set, $A = I$, a homogeneous structure is developing in the center, as it is expected due to the high symmetry of the image, see Fig. 1(d).

A impression of the denoising and gap-closing capabilities can be obtained from the results in Fig. 2. The noisy version of a artificial data set with a grid-like is CED-filtered. As before the directional preferences are conveyed by the reduction matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for the x-direction and $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for the y-direction. The noise is removed and the lines in the selected direction are getting completed.

Fig. 3 shows the results of directionally selective CED-filtering if the direction of the coherent structure (here the y-direction) does *not* coincide with the direction of the ellipsoids (here the x-direction). By selecting the x-direction with a proper choice of A we allow for a the enhancement of coherent structures, that is, the closing of lines if there is a change in this

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x-direction. In Fig. 3(b) we have such a change, hence the present lines in y-direction are completed. We do not have a change in y-direction (the balls and ellipsoids have the same y-extension), hence the selection of the y-direction via $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ triggers no enhancement at all. Therefore this directional CED-filtering has no effect, see Fig. 3(c).

The last experiment depicted in figure 4 demonstrates that even areas with no information may constitute a coherent structure. A matrix field with a visually diverging structure CED-filtered without directional preferences. The (almost) empty lines in x-direction are getting filled while the two lines in y-direction remain untouched by the filtering. The explanation is that in x-direction we have changes in the shape and orientation of the ellipsoids in the vicinity of the empty lines, hence, the gap-closing quality of CED-filtering is coming into effect. However, proceeding in y direction no changes of the surrounding ellipsoids is discernable, rendering the CED-filtering idle in this direction.

7   Summary and Future Work

In this paper we have developed a novel structure tensor concept for matrix fields. This approach is based on an operator-algebraic view on matrices and their rich algebraic properties. We have shown how to infer directional information from this high dimensional data by specifying directional preferences, and clarified Weickert's structure tensor for matrix fields as a special case of our extended concept. An application of the new tensor concept enabled us to develop a directionally selective coherence-enhancing diffusion filtering of matrix fields by employing a generic differential calculus framework for matrices. The matrix-valued CED-filtering exhibits similar behaviour as its scalar counterpart.

Current work encompasses the investigation of further opportunities to steer the filtering process,e.g. in the extension step, and its relation to other customisable tensor concepts for matrix fields. Future research will focus on further applications of the extended structure tensor concepts in image processing for matrix fields.

Acknowledgments
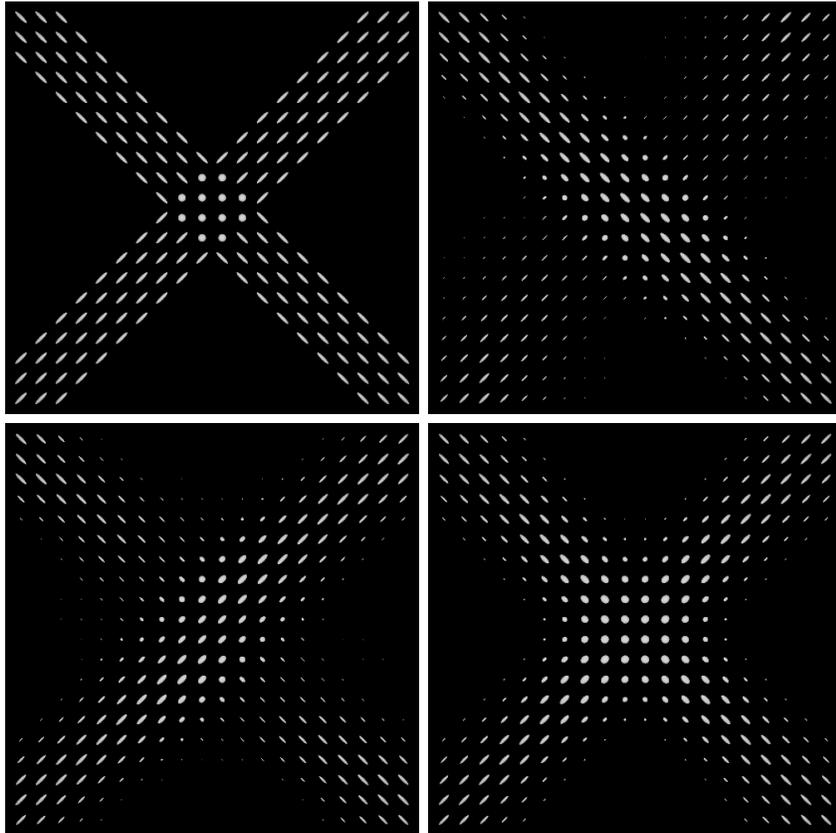
Figure 1: **(a) Top left:** Artificial data set of ellipsoids indicating a crossing. **(b) Top right:** Effect of coherence-enhancing filtering if preference is given to the direction $(1, -1)$ **(c) Bottom left:** Effect of coherence-enhancing filtering if preference is given to the direction $(1, 1)$. **(d) Bottom right:** Effect of coherence-enhancing filtering if no directional priority is established.
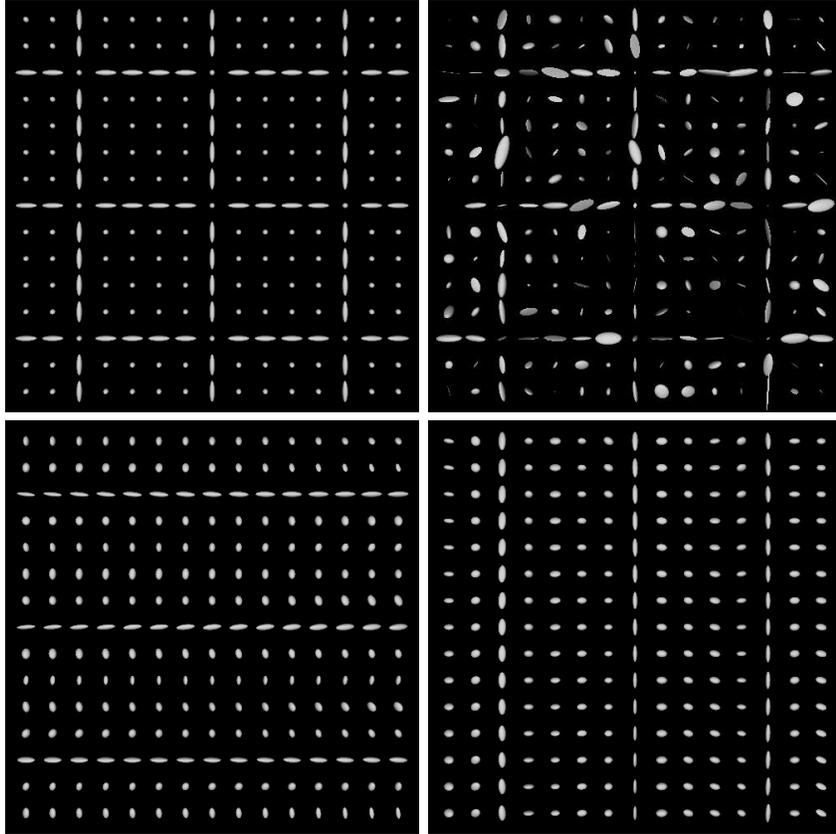
Figure 2: **(a) Top left:** Original matrix field with grid-like structure. **(b) Top right:** (a) polluted with truncated Gaussian noise in the eigenvalues while the orthogonal matrices result from three Euler matrices with uniformly distributed angles.
(c) Bottom left: Result of CED-filtering with preference on the horizontal x-direction. **(d) Bottom right:** The same but with preference on the vertical y-direction.
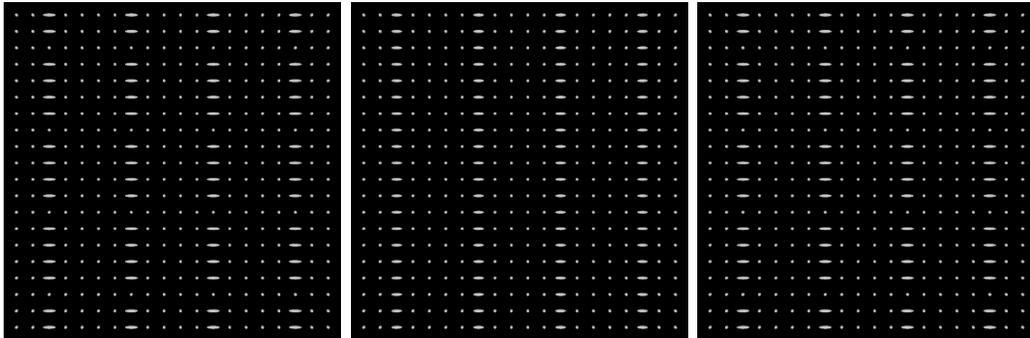
Figure 3: **(a) Left:** Original image with coherent structure in y-direction and ellipsoids pointing in x-direction. **(b) Middle:** Result of CED-filtering with preference on the horizontal x-direction. **(c) Right:** Result of CED-filtering with preference on the horizontal y-direction.
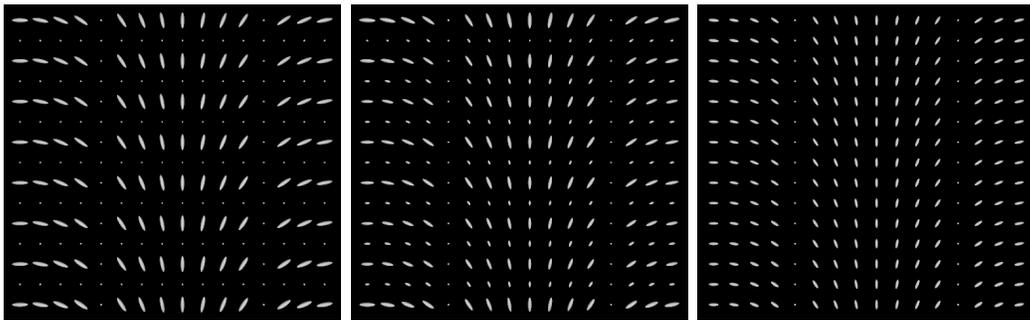
Figure 4: **(a) Left:** Artificial incomplete coherent structure. **(b) Middle:** After CED-filtering with stopping time $t = 0.3$. **(c) Right:** After CED-filtering with stopping time $t = 3$
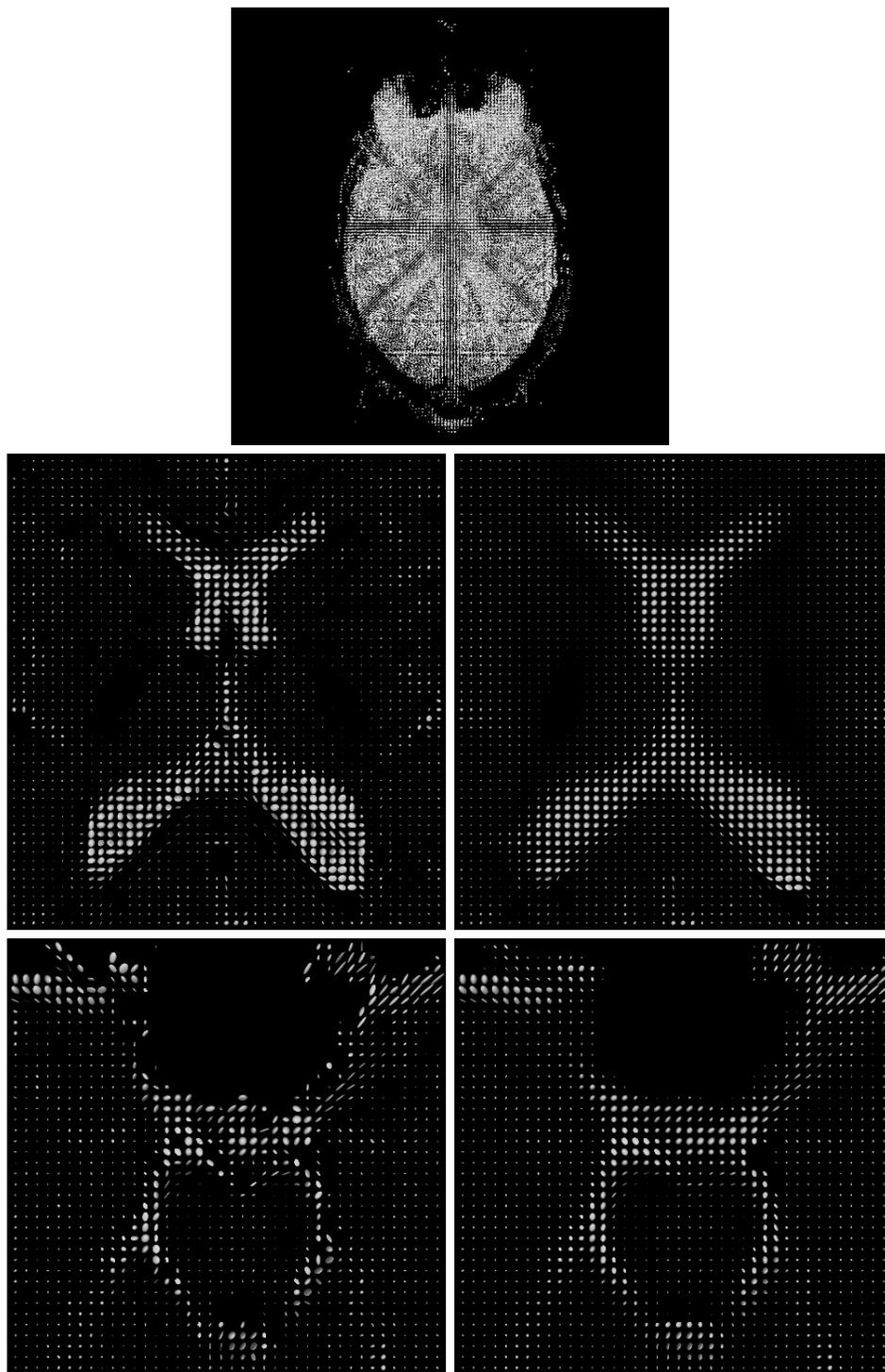
Figure 5: **(a) Top:** Original DT-MRI data set of a human head, $128 \times 128 \times 30$ voxels. **(b) Left column:** 2-D tensor fields extracted from this data set, $45 \times 53 \times 1$ voxels. **(c) Right column:** Effect of coherence-enhancing diffusion filtering.

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us with the DT-MRI data set and for discussing questions concerning data conversion.

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