The Bessel Scale-Space

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Abstract. In this paper we propose a novel type of scales-spaces which is emerging from the family of inhomogeneous pseudodifferential equations $(I - \tau\Delta)^{\frac{t}{2}}u = f$ with $\tau \geq 0$ and scale parameter $t \geq 0$. Since they are connected to the convolution semi-group of Bessel potentials we call the associated operators $\{R^n_{t,\tau} \,|\, 0 \leq \tau, t\}$ either Bessel scale-space $(\tau = 1)$, R^n_t for short, or scaled Bessel scale-space $(\tau \neq 1)$. This is the first concrete example of a family of scale-spaces that is **not** originating from a PDE of parabolic type and where the Fourier transforms $\mathcal{F}(R^n_{t,\tau})$ do **not** have exponential form. These properties make them different from other scale-spaces considered so far in the literature in this field.

In contrast to the α-scale-spaces the integral kernels for $R^n_{t,\tau}$ can be given in explicit form for any $t, \tau \geq 0$ involving the modified Bessel functions of third kind K_ν. In theoretical investigations and numerical experiments on 1D and 2D data we compare this new scale-space with the classical Gaussian one.

Keywords: Bessel potential, Bessel-functions, α-scale-space, convolution, semi-group, pseudodifferential operator,co-histogram.

1 Introduction

In retrospect modern scale-space theory began with the pioneering work of Taizo Iijima [17] in the late fifties. Although his work was not noticed by the western scientific community for decades the vivid research on scale-space methodologies has resulted in a large amount of techniques valuable for image processing and computer vision. This is documented in numerous articles and books, see [12,31,21,29,33] and the literature cited therein.

The Gaussian scale-space is the archetype of a linear scale-space. Its relation to linear diffusion processes was first pointed out to the image processing community by Iijima [18].

However, scale-space properties can also be spotted in non-linear diffusion processes, a field inspired by the path-breaking work of Perona and Malik [26]. These non-linear theories embrace anisotropic diffusion processes [33,27], morphological operations [32,6,19] as well as the evolution of level curves [2,24,28,20].

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Highly non-linear, sometimes even degenerated differential equations are the mathematical language to describe these theories [31,33,15,3,13,8].

Be that as it may, the linear setting, meaning the assumed validity of the superposition principle, and the exploration of underlying axiomatic theory was and is an active field of research, [4,2,33,12,22,25,34] and [10].

In this linear setting the importance of the Gaussian scale-space cannot be overestimated, although in recent years other concrete examples of linear scale-space concepts have received considerable attention:

- First the Poisson scale-space arising from the Laplace equation in potential theory has been introduced by Felsberg and Sommer [11] to image processing. It allows an explicit analytical integral representation with the Poisson kernel.
- After that the so-called α-scale-spaces with $\alpha \in]0,1]$ have been proposed as the continuous link between the trivial ($\alpha = 0$), the Poissonian ($\alpha = \frac{1}{2}$) and the Gaussian ($\alpha = 1$) scale-space. They are ruled by an pseudodifferential equations, and unfortunately no exact integral representation formulas for their solutions are known. See [10] for a very comprehensive exposition about theory and history of this scale-space family.
- Very recently the relativistic scale-spaces [7] instigated by a Schroedinger pseudodifferential equation from theoretical physics have been shown to bridge the gap between Poisson (‘zero-mass-limit‘) and the trivial scale-space (‘infinite-mass-limit‘). Explicit integral formulas involving kernels with Bessel functions of the third kind have been given in [7].

All these examples have in common that they emanate from (pseudo-) differential equations of parabolic type, such as the $\alpha-$scale-spaces:

$$\partial_t u = -(-\Delta)^\alpha u$$

with initial condition $u(x,0) = f(x)$.

The goal of this paper is to investigate the scale-space that arise from the following inhomogeneous elliptic PDE involving arbitrary positive powers $t \geq 0$ of the Laplacian and the identity operator I:

$$(I - \Delta)^{\frac{t}{2}} := \left(I - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \right)^{\frac{t}{2}} = f, \tag{1}$$

with a suitable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

The parameter t should be interpreted as a smoothing parameter: The application of an partial differential operator to a function u roughens it. Intuitively, if u is to fulfill (1) (even in the distributional sense) it must be smooth enough to produce f, and the larger t is the smoother the function u has to be. Hence, solving (1) for u means in effect calculating smoother versions of f.

However, equation (1) is *not* an evolution equation of parabolic type. Although it is not done in this article, one has the opportunity to tackle this equation with the highly developed numerical methods for elliptic PDEs. Furthermore, inhomogeneous PDEs might be the starting point for a fruitful nonlinear and anisotropic theory, just as it was the case for the Gaussian scale-space. We will examine the smoothing procedure ruled by (1), establish the associated convolution semi-group properties by spectral methods. In contrast to the scale-space examples mentioned above this semi-group is not of exponential type.

The associated integral representation kernels are explicitly known as Bessel potentials, a generalisation of Riesz potentials. Hence, the properties of this scale-space can be explored also with methods from real analysis.

The paper is structured as follows: In the following section we use the Fourier transform a function $f \in L^2(\mathbb{R}^n)$ given by

$$\mathcal{F}(f)(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} f(x)\, dx\,.$$

to study (1). This will lead directly to the definition of the Bessel scale-space. After a study of its properties we will also present scaled versions of the Bessel scale-space. Experiments illustrating the potential and limitations of the novel scale-spaces are described in Section 3. A summary and an outlook for ongoing in Section 4 complete the paper.

2 Bessel Scale-Space

We recall that the action of the differential operator Δ is multiplication by $-4\pi|k|^2$, implying that (1) Fourier transforms into

$$(1 + 4\pi^2|k|^2)^{\frac{t}{2}}\, \hat{u} = \hat{f}\,.$$

According to theory of spectral methods for PDEs this entails that formally the solutions to (1) are computed via convolution with the integral kernel $G_n(\cdot, t)$ which appears as the inverse Fourier transform of

$$\mathcal{F}(G_n)(\cdot, t) := \frac{1}{(1 + 4\pi^2|k|^2)^{\frac{t}{2}}}\,,$$

that is,

$$G_n(x, t) = \int_{\mathbb{R}^n} \frac{1}{(1 + 4\pi^2|k|^2)^{\frac{t}{2}}} e^{2\pi i k \cdot (x-y)}\, dk\,.$$

This integral can be evaluated in every dimension n yielding the known explicit formula for the Bessel kernels [23,9]

$$G_n(x, t) = \frac{1}{\sqrt{\pi}^{\,n} \sqrt{2}^{\,n+t-2}\, \Gamma(\frac{t}{2})} \frac{K_{\frac{n-t}{2}}(|x|)}{|x|^{\frac{n-t}{2}}}\,,$$

Γ denotes the Gamma function and K_ν stands for the modified Bessel function of third kind with index ν.

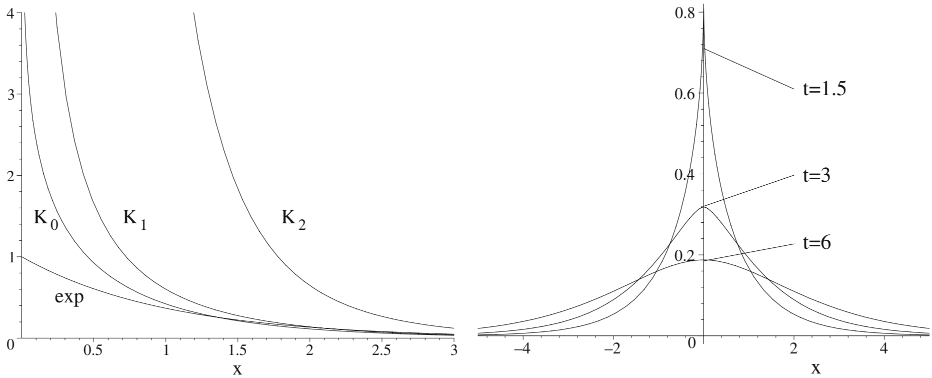

Fig. 1. *Left:* Comparison of the exponential and Bessel functions K_0, K_1 and K_2. *Right:* Examples of the Bessel kernel for $n = 1$ with $t = 1.5$, 3, 6.

The Bessel functions K_ν can be evaluated via fast converging series expansions and three-term recursive formulas. For more details see [1]. Figure 1 compares the exponential function e^{-x} with some Bessel functions. The Bessel functions are exponentially decaying for large x.

Using formulas in [1] for K_ν one can derive explicit expressions of Bessel kernels for special values of t:

$$G_n(x, n+1) := \frac{1}{\pi^{\frac{n}{2}} 2^n \Gamma(\frac{n+1}{2})} \, e^{-|x|},$$

which is a continuous function, not differentiable at $x = 0$, and

$$G_n(x, n+3) := \frac{1}{\pi^{\frac{n}{2}} 2^{n+1} \Gamma(\frac{n+3}{2})} \, (1 + |x|) \, e^{-|x|},$$

which is in fact twice continuous differentiable in $\mathbb{R}^n$.

This has an interesting effect: a merely continuous function convolved with $G_n(x, n+k)$ produces only a C^{k-1}-smoothed version. This behaviour is different from Gaussian, Poissonian, or relativistic scale-spaces, where the filtered functions are even analytical for every scale parameter $t > 0$.

Figure 2 displays the Bessel kernel for various values of t and also its comparison with a Poisson and a Gaussian kernel.

For notational convenience we define the operator R_t^n on $L^2(\mathbb{R}^n)$ via the convolution

$$R_t^n f(x) := (G_n(\cdot, t) * f) \; (x) = \int_{\mathbb{R}^n} G_n(x - y, t) f(y) \, dy \,. \tag{2}$$

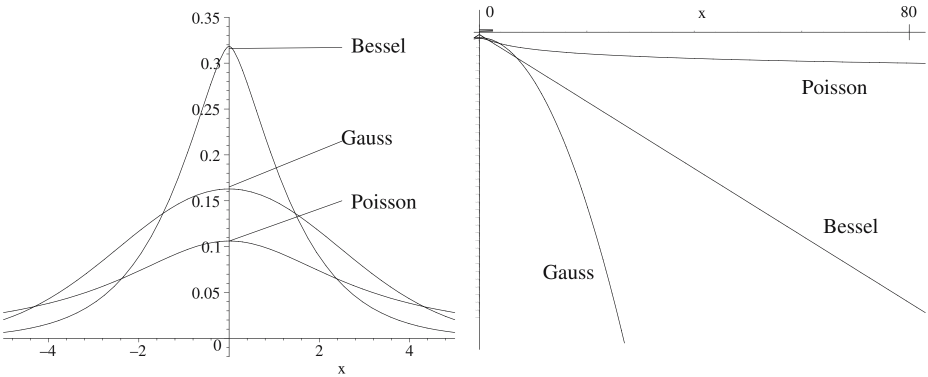

Fig. 2. *Left:* Comparison between different kernels including Bessel ($\tau = 1$), Gaussian, and Poisson kernel in 1D centered at the origin with $t = 3$. *Right:* Comparison of the asymptotic behaviour of the same kernels for large values of x (logarithmic scale on y-axis).

2.1 Behaviour of $G_n(\cdot, t)$ in the Limit $t \downarrow 0$

According to a theorem of P. Levi [5] stating the continuity of the (inverse) Fourier transform the relation

$$\mathcal{F}(R_t^n)(k) = \frac{1}{(1 + 4\pi|k|^2)^{\frac{t}{2}}} \longrightarrow 1 \quad \text{if } t \downarrow 0$$

confirms that R_t^n approximates the identity operator I in the distributional sense if t is small. This can also be shown by methods from real analysis based on the explicit knowledge of the Bessel potentials.

2.2 Semigroup Properties

From the theory of contraction semi-groups [16] we infer that the operator R_t^n determines a contraction semi-group on $L^2(\mathbb{R}^n)$. Indeed, in view of Plancherel's theorem, it is enough to verify that the Fourier transforms $\mathcal{F}(R_t^n)$ of the family $\{R_t^n\}$ satisfy the conditions

1. $\mathcal{F}(R_{s+t}^n)\mathcal{F}(f) = \mathcal{F}(R)_s^n \mathcal{F}(R_t^n)\mathcal{F}(f) = \mathcal{F}(R_t^n)\mathcal{F}(R_s^n)\mathcal{F}(f)$ for all $s, t \geq 0$,
2. $\|\mathcal{F}(R_t^n)\mathcal{F}(f) - \mathcal{F}(R_s^n)\mathcal{F}(f)\|_2 \longrightarrow 0 \quad \text{for} \quad t \longrightarrow s$,
3. $\mathcal{F}(R_0^n) = 1$, expressing the fact that $R_0^n = I$, the identity,
4. $\|\mathcal{F}(R_t^n)\mathcal{F}(f)\|_2 \leq \|\mathcal{F}(f)\|_2$, the contraction property.

Due to the properties of the elementary functions $\frac{1}{(\sqrt{1+c})^t}$ with $c > 0$ it is not difficult to check that the operator R_t^n indeed fulfills these conditions.

2.3 Regularity

We define the Sobolev spaces $H^s(\mathbb{R}^n)$ as in [30] (with $2\pi k$ instead of k) by

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \mid \left(1 + 4\pi^2|k|^2\right)^{\frac{s}{2}} \mathcal{F}(u) \in L^2(\mathbb{R}^n) \right\}$$

for all functions in $L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}$.
Then it follows without difficulty that R_t^n increases the regularity:

$$R_t^n : H^s(\mathbb{R}^n) \longrightarrow H^{s+t}(\mathbb{R}^n)$$

In this sense the operator indeed produces smoother versions $\tilde{u}^t = R_t^n f$ of a given $f \in L^2(\mathbb{R}^n)$. Summarising the analysis above we state

Proposition 2.1. *1. The families of operators $\{R_t^n \mid t \geq 0\}$ form an* additive semi-group *for any fixed $n \geq 0$.*
2. For every $t \geq 0$ the average grey-value is preserved under the action of R_t^n.
3. The operators R_t^n are translational invariant.

However, it is not difficult to see that the Bessel scale-space is not scale invariant. As already indicated before, the scale parameter t plays also the role of a smoothing parameter; roughly speaking, the smoothness is increased by t. This is not the case for the standard linear scale spaces, where the smoothness of the filtered signal immediately jumps to its highest level, analyticity.

2.4 Scaled Bessel Scale-Spaces

The following generalisation of the Bessel kernel is close at hand: we introduce a scaling parameter $\tau \geq 0$ via

$$G_{t,\tau}^n(x) := \tau^n G_t^n(\tau x).$$

Then we have

$$\mathcal{F}(G_{t,\tau}^n)(k) = \frac{1}{(1 + 4\pi^2\tau^2|k|^2)^{\frac{t}{2}}},$$

furthermore, all the properties of G_t^n mentioned above carry over, essentially verbatim, to $G_{t,\tau}^n$, including semi-group, contraction and limit properties. $R_{t,\tau}^n$ denotes the corresponding convolution operator. For $\tau = 0$ the operator degenerates to the identity, $R_{t,0}^n = I$, while for $\tau = 1$ we obtain the Bessel scale-space, $R_{t,1}^n = R_t^n$.

Numerical examples for these scaled versions of the Bessel scale-space are presented in the following experimental section.

3 Numerical Experiments

In this section we display some results of numerical experiments to visualise the properties of the Bessel and the scaled Bessel scale-spaces. We contrast the novel

Bessel with the Gaussian scale-space. First we take a look at the Bessel and Gaussian scale-space in 1D. The results are captured in a 3D-plot in Fig. 3. We have chosen a signal with discontinuities to visualise the regularising properties of the Bessel scale-space. The differences are not dramatic, especially since the weaker regularity of the Bessel-filtered signals is not discernable from the analyticity of the Gaussian filtered results. In order to compare the effect of Gauss and Bessel filtering of 2D-images we utilised so-called co-histograms [14]. Co-histograms $h_{f,g}(m,n)$ are 2D-histograms encoding the frequency of *ordered pairs* of grey values (m,n) of an image pair (f,g). They are constructed via the formula

$$h_{f,g}(k,l) = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} \delta(f_{i,j}, k) \cdot \delta(g_{i,j}, l)\,,$$

where δ stands for the Kronecker symbol and $M \times N$ is the size of the images f, g. Figure 4 depicts the co-histogram as a grey value image. Differences in the images f and g result in asymmetry of the co-histogram and its departure from being diagonal. At the very beginning Gauss and Bessel filtering of the office image without noise do not yet have a strong effect ($t = 0.1$), hence the diagonal dominant form of the co-histogram. The appearance changes with increasing scale t, furthermore, in the limit $t \to \infty$ the co-histogram will tend towards one bright spot on the diagonal marking the average grey value common to both filter processes. For larger times there is no visible difference in the ability of removing (Gaussian) noise between the two scale-space concepts. Only for very small times there is a discrepancy indicated by the spread of the corresponding co-histogram (Fig. 4, middle column, second row).

The situation is different for a binary image (last column of Fig. 4); in this case the co-histograms indicate a clearly discernable difference between the two types of filtering throughout the evolution processes.

We remark that fixing the parameter t and using τ as parameter also leads to a scale-space structure, referred to as the scaled Bessel scale-space in the previous chapter. Fig. 5 contrasts a scaled version (right column) with the non-scaled version of the Bessel scale-space. One may notice the convergence towards the mean value for increasing values of t or τ, respectively.

4 Conclusion

The goal of this paper is to introduce the novel two-parameter family of Bessel scale-spaces. In proposing this peculiar example we hope to convey our opinion that not only parabolic (pseudo-)differential equations can serve as a birthplace for scale-spaces. The underlying Bessel convolution semi-group turned out to possess a non-exponential Fourier transform. The degree of smoothness of the filtered data grows steadily (in terms of Sobolev exponents) for increasing scale parameter t, in contrast to other common scale-spaces. Nevertheless, opposite to the α-scale-spaces these new scale-spaces admit integral representations with

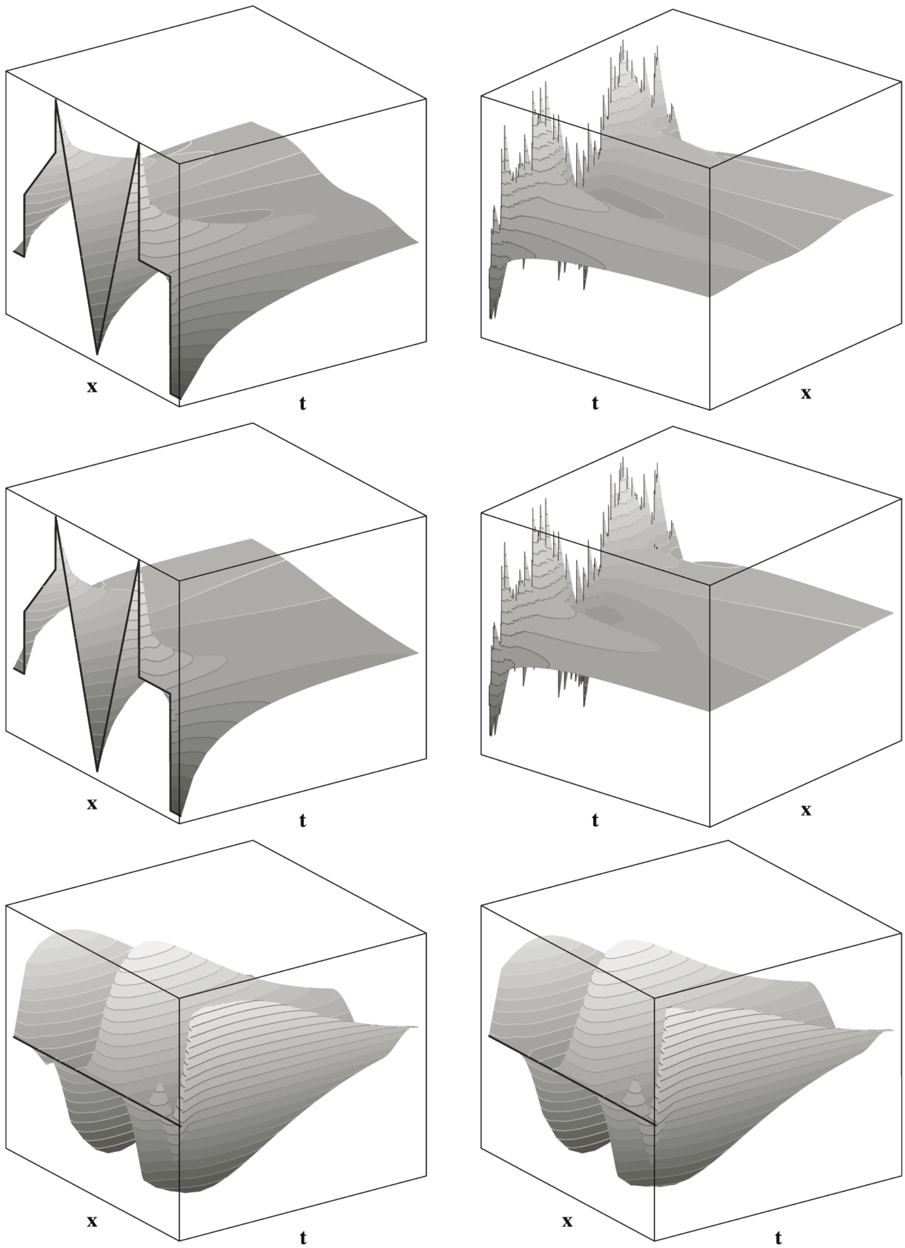

Fig. 3. Bessel scale-space in 1D. *Left column, top :* Smoothing of a signal in Bessel scale-space. *Left column, middle :* Smoothing of this signal in Gaussian scale-space. *Left column, bottom :* Difference in signal evolution w.r.t. scale-spaces above. Note that the scale on the z-axis has been stretched by the factor 7 in comparison with the images above. *Right column:* The same with noisy signal (Gaussian noise added to the original signal on the left).

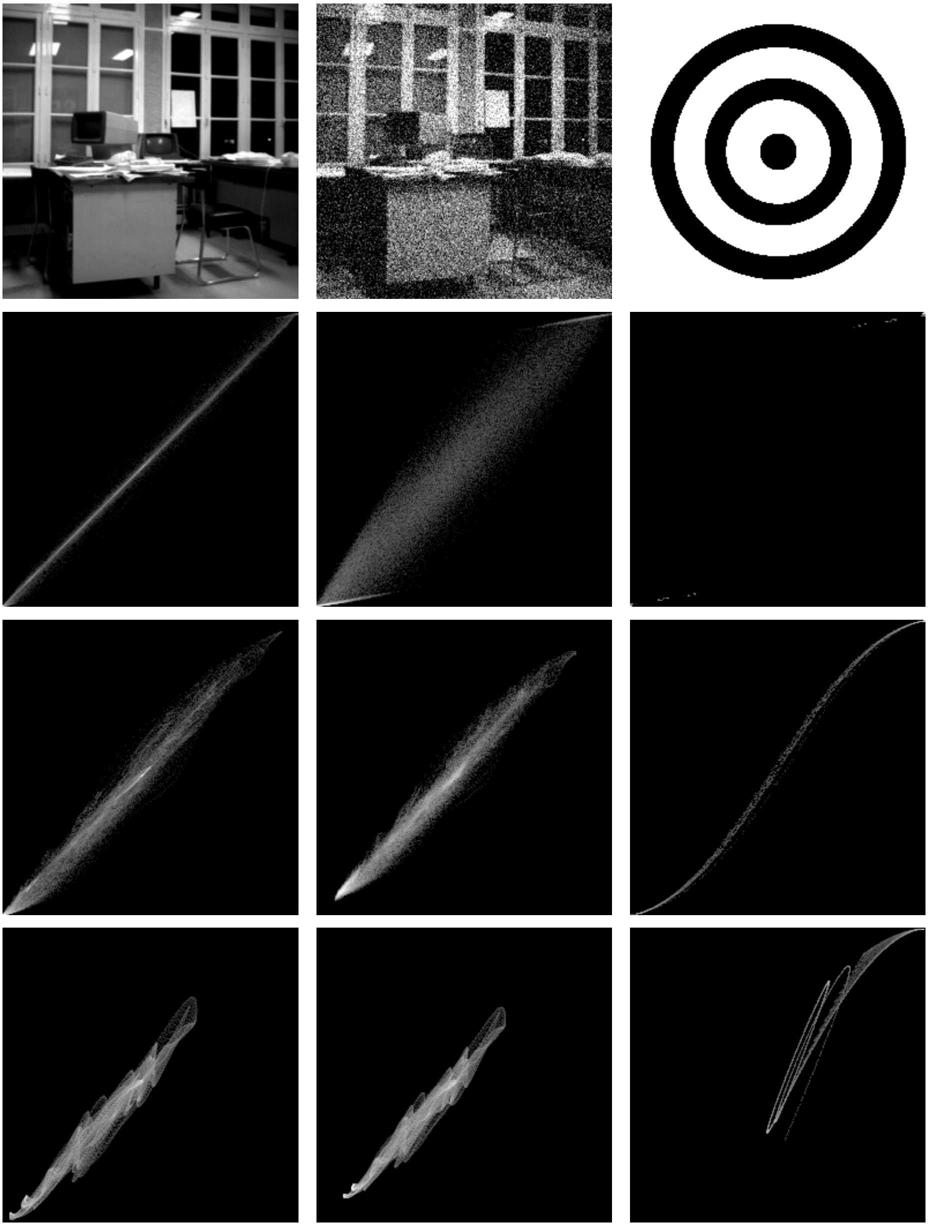

Fig. 4. Co-histograms: Comparison of Gaussian and Bessel scale-space in 2D. *Top row:* Original images. *Second row:* Co-histograms comparing Gauss and Bessel filtering of the corresponding images of the first row with $t = 0.1$. *Third row:* The same with $t = 10$. *Fourth row:* The same with $t = 100$.

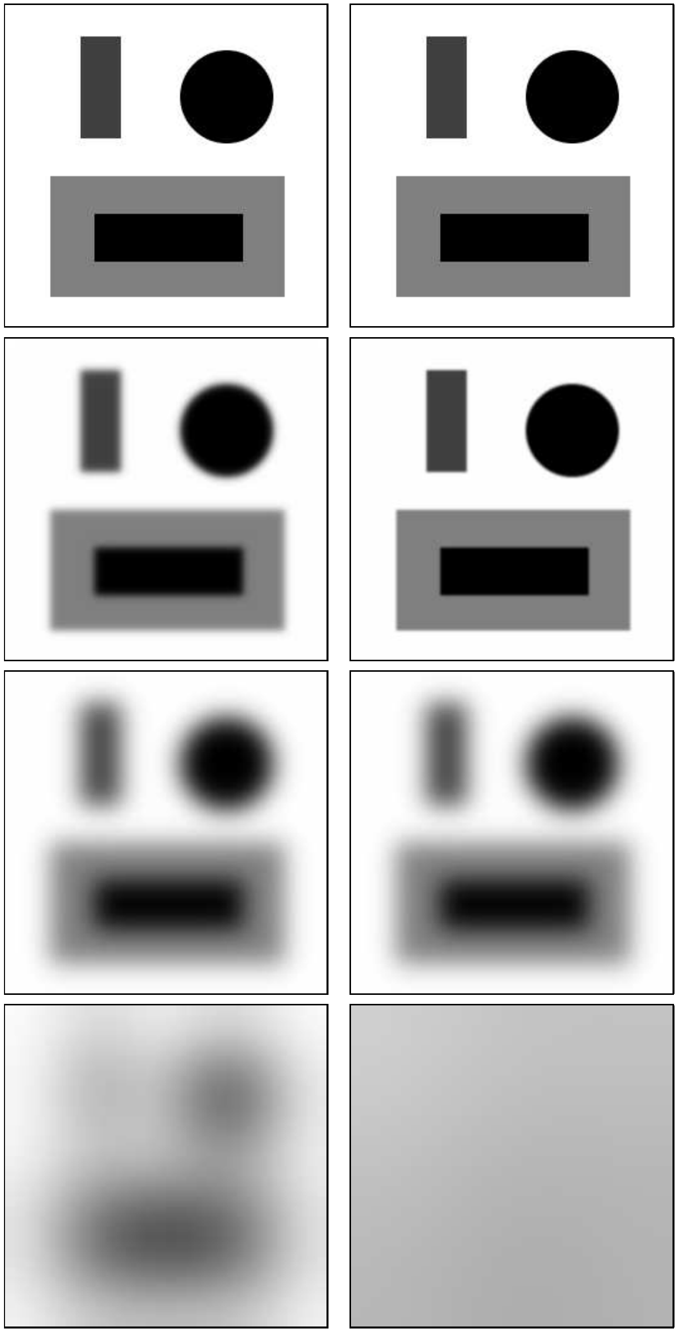

Fig. 5. Comparing non-scaled and scaled Bessel scale-spaces in 2D. *Left column:* Non-scaled Bessel scale-space, $\tau = 1$. *Left column, from top to bottom:* $t = 0$, 10, 100, 1000. *Right column:* Scaled Bessel scale-space, with fixed $t = 100$. *Right column, from top to bottom:* $\tau = 0$, 0.1, 1, 10.

explicitly known kernels. They involve modified Bessel functions K_ν of the third kind and hence bear some resemblance to the relativistic scale-spaces.

Ongoing research on Bessel scale-spaces encompasses studies of variational formulations, special features as well as non-linear extensions and their numerical treatment.

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