SHAPE FROM SHADING WITH SPECULAR HIGHLIGHTS: ANALYSIS OF THE PHONG MODEL

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ABSTRACT

Recently, non-Lambertian models for perspective shape from shading (PSFS) have received attention in the literature. The Phong-based PSFS model combining Lambertian and specular reflection has been shown to give good results for objects in real-world images. In this paper, we present at hand of this model the first analysis of a non-Lambertian PSFS model in the literature. We show mathematically and experimentally how crucial the specular part of the model is for the reconstruction. Moreover, we give a detailed analysis of the Hamiltonian defining the model. While the non-Lambertian reflectance is generally assumed to lead to numerical difficulties, this investigation shows that an efficient fast marching scheme can be applied successfully without losing depth information. Our work represents a first step towards the thorough understanding of non-Lambertian PSFS models.

***Index Terms*—** shape from shading, perspective projection, Phong reflectance, non-convex Hamiltonian, fast marching method

1. INTRODUCTION

Shape from Shading (SFS) is a fundamental problem in computer vision. Given information about the illumination and the reflection properties in an imaged scene, the SFS task is to compute the 3-D shape of depicted objects. Just a few years ago, SFS was understood as a severely ill-posed problem [1]. Then *perspective SFS (PSFS)* models have been proposed by several authors [2–4] which deliver substantially better results and are even well-posed to some degree, cf. [5] for an exposition. A number of successive works were concerned with PSFS for objects with Lambertian light reflectance. However, as this is not a realistic reflectance model, the Lambertian PSFS approach has been extended to non-Lambertian settings, incorporating *specular highlights* in the scene [6, 7].

We deal in this work with the non-Lambertian model described in [7]. It includes specular reflectance by making use of the classic *Phong model* from computer graphics [9]. In [8, 10] it has been shown that this model enables SFS for realistic grey value input images. Even though it is a relatively simple prototype for PSFS with non-Lambertian reflectance, this model is up to now not well-understood.

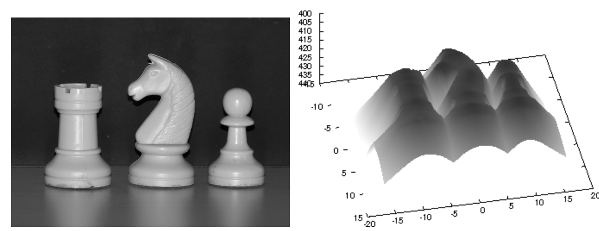

Fig. 1. Perspective Phong-based SFS for a real-world input image (figure adopted from [8]). *Left.* Original image. *Right.* 3-D reconstruction. What is the crucial mechanism in the model that enables such good results for the chess figures?

Our contribution. In this paper, we give the first detailed analysis of a non-Lambertian PSFS model in the literature. We show that the specular reflection terms of perspective Phong-based SFS (P^2SFS) have crucial influence at *singular points* (see e.g. [11]) that largely determine the shape computation. The analytic investigation is complemented by an experiment which shows that even the shape of objects with very strong specular reflection (visually close to a glossy plastic surface) can be reconstructed correctly. In addition, an in-depth analysis of the non-convexity of the model is presented. Our results indicate that the efficient fast marching (FM) method can be used without losing depth information at and around singular points. This not only shows that the purely algorithmic FM approach proposed in [8] is correct, it also implies that one may consider the FM method for use with other non-Lambertian PSFS models.

In what follows, we summarise the P^2SFS model. We proceed with the analysis of singular points and an exposition on non-convexity, followed by a numerical test and a conclusion.

2. PSFS WITH PHONG-REFLECTANCE

We now briefly recall the P^2SFS model developed in [7]. Let $(x, y)^\top \in \mathbb{R}^2$ be in the image domain Ω. Furthermore: *(i)* $u := u(x, y)$ denotes the unknown 3-D depth; *(ii)* $I := I(x, y)$ is the normalised image brightness; *(iii)* f is the focal length denoting the distance between the optical centre of the camera and the 2-D image plane.

Clearly $u > 0$ holds, since the depicted scene is in front of the camera, and the depth u is measured in terms of multiples of f. In order to simplify the constituting equation, we will use $v := \ln(u)$ as unknown variable.

With these definitions at hand and assuming that the light source is positioned at the optical centre of the camera, the constituting equation of the P^2SFS model is given by the *Hamilton-Jacobi equation (HJE)*

$$\frac{\mathsf{f}^2 W}{Q}\left(I - k_a I_a\right) - k_d I_d e^{-2v} - \frac{W k_s I_s e^{-2v}}{Q} R^\alpha = 0 \quad (1)$$

$$\text{with} \quad W := \sqrt{\mathsf{f}^2 |\nabla v|^2 + (\nabla v \cdot (x, y)^\top)^2 + Q^2} \quad (2)$$

$$Q := \mathsf{f} / \left(\sqrt{x^2 + y^2 + \mathsf{f}^2}\right) \quad (3)$$

$$R := 2Q^2 / W^2 - 1 \quad (4)$$

Thereby, $\nabla v = (v_x, v_y)^\top$ is the gradient of v, and the lower indices in v_x and v_y denote partial derivatives. The HJE (1) is a *partial differential equation (PDE)* whose solution is to be understood in a viscosity sense, cf. [5] for related discussions. The PDE (1) needs to be complemented by boundary conditions ensuring that no shape information propagates into the image domain from outside.

For the P^2SFS model the surface reflectance is described by the Phong-model [9]. The brightness equation for one light source and grey-value images that is the origin of (1) reads as

$$I = k_a I_a + \frac{1}{r^2}\left(k_d I_d \cos\phi + k_s I_s (\cos\theta)^\alpha\right) \quad (5)$$

Here I_a, I_d, and I_s are the intensities of ambient, diffuse, and specular components of light, respectively. The constants k_a, k_d, and k_s with $k_a + k_d + k_s \leq 1$ denote the ratio of ambient, diffuse, and specular reflection. The ambient light is light present everywhere in a given scene. The intensity of diffusely reflected light in each direction is proportional to the cosine of the angle ϕ between surface normal and light source direction. The amount of specular light reflected towards the viewer is proportional to $(\cos\theta)^\alpha$, where θ is the angle between the ideal mirror reflection direction of the incoming light and the viewer direction. Let us note that the expression $(\cos\theta)^\alpha$ is replaced by zero if the cosine evaluates to a negative number. The parameter α models the roughness of the material. The number $r := r(x, y)$ is the distance between the light source and the object surface. In case where the light source coincides with the optical centre, then $r = \mathsf{f}u$; $1/r^2$ is the inverse square law of light attenuation.

The simplicity of this reflectance model, compared to other approaches, is a reason why the P^2SFS model can be considered as a prototype for non-Lambertian PSFS models.

3. ANALYSIS OF SINGULAR POINTS

We will now investigate the HJE (1) with respect to the importance of contributing terms at singular points. As defined by Oliensis [11], these are the points with the highest brightness in the image.

Let us consider now the points of minimal distance between the unknown surface and the camera. Since the light source is positioned at the optical centre of the camera, the angles ϕ and θ in the brightness equation (5) will be zero there, i.e. evaluating the cosine we obtain the maximal value 1. Moreover, because of the light attenuation term $1/r^2$, points sharing this property but that are at a larger distance from the camera do not feature the maximal brightness.

To summarise, in the P^2SFS model the singular points also define the points of minimal distance between object and camera (excluding artificial settings like e.g. that the photographed scene is a ball of uniform radius around the camera). Therefore, it also holds $\nabla v = 0$ since this is a necessary condition for having a local depth extremum.

Let us also note that the light attenuation term combined with setting the light source at the optical centre serves to avoid the classic convex/concave ambiguity, cf. the discussion in [5]. Thus, in the P^2SFS model singular points determine in most situations of practical interest uniquely the shape of objects.

The HJE at singular points. Concerning the setting at singular points, let us make use of $\nabla v = 0$ to simplify the P^2SFS equation. Plugging this term into the equation gives

$$\mathsf{f}^2\left(I - k_a I_a\right) = k_d I_d e^{-2v} + k_s I_s e^{-2v} \quad (6)$$

Obviously, the specular term is in general as important as the diffuse term. For $k_s > k_d$ as it can be the case for strongly reflecting materials, specular reflectance largely determines the depth at singular points.

In addition to this point of view, the understanding of how a model performs at singular points is crucial for the construction of numerical methods. This can be seen as follows. Any discretisation $\hat{\nabla} v$ of the partial derivatives in ∇v that results in $\hat{\nabla} v = (0, 0)^\top$ at local extrema should lead to the exact computation of the depth as in (6). This holds for instance for the discretisations investigated in [12] for the Lambertian PSFS model. Considering iterative schemes, this means that all the iterative solutions are fixed at singular points. An important alternative to iterative solvers is the FM method. Let us stress that this method needs an estimate of the depth at singular points as initialisation. The solution returned by the FM scheme depends on these initial values.

The HJE away from singular points. In order to clarify the role of specular terms, it is useful to distinguish the situation for singular points from the case when the HJE is evaluated for $\nabla v \neq (0,0)^\top$. While it is clear that the specular term is strong at specular reflections, the question arises *how much* it influences the PDE model away from specular highlights. This investigation will also help to answer the question, when exactly the model switches between non-convexity at highlights and convexity where the Lambertian part dominates.

To this end we investigate the size of the numbers appearing within the HJE and its discretisations. On a discrete level, let us generally set $h := 1$, where h represents the characteristic grid size of the input image. With $h = 1$, the numbers appearing in the vector $(x,y)^\top$ are directly related to the numbers of pixels in the image. Assuming that an object of interest for the reconstruction is in the centre of the input image, the interesting range of $\left|(x,y)^\top\right|$ will be in the order of up to a few hundred, even for large images.

Generally, there is also an underlying relation $\mathsf{f} \sim h$. In the setting specified up to now (for $h = 1$), notably the number used for the focal length f will in general be in the range 10^2 to 10^4. As a consequence, it seems interesting to explore the orders of magnitude with respect to f within the terms of the HJE (1). With

$$W = \mathsf{f}\sqrt{|\nabla v|^2 + (\nabla v \cdot (x,y)^\top)^2/\mathsf{f}^2 + Q^2/\mathsf{f}^2} = \mathcal{O}(\mathsf{f}) \quad (7)$$

and

$$Q = 1/\sqrt{x^2/\mathsf{f}^2 + y^2/\mathsf{f}^2 + 1} = \mathcal{O}(1) \quad (8)$$

since any occuring term divided by f is small, we obtain in leading order:

$$\text{HJE main part:} \quad \frac{\mathsf{f}^2 W}{Q}\,(I - k_a I_a) = \mathcal{O}(\mathsf{f}^3) \quad (9)$$

$$\text{Diffuse source term:} \quad k_d I_d e^{-2v} = \mathcal{O}(1) \quad (10)$$

$$\text{Specular term:} \quad \frac{W k_s I_s e^{-2v}}{Q} R^\alpha = \mathcal{O}(\mathsf{f}) \quad (11)$$

To get the complete picture, let us stress that the term R is equal to zero for

$$W > \sqrt{2}Q \quad (12)$$

One can easily see that the values of Q can be expected in the range of 0.7 to 1 in the centre region of an image where the object of interest should be located. Simplifying for the moment W to $\mathsf{f}\,|\nabla v|$, we see that R will be only non-zero for very flat gradients close to singular points.

As a consequence, we observe that the Lambertian part of the model determines the reconstruction of the shape away from singular points. We also observe another mechanism in the part dominated by the Lambertian model: Variations of the depth in the source term – as arising e.g. during an update step within an iterative solver, cf. [5] – will lead to very small changes in $\hat{\nabla} v$ since these are multiplied essentially by f^3. This makes the model computationally robust.

Applying the same logic at singular points, see (6), we observe the contrary. Given data I are multiplied by f^2, so that there is a delicate balance between the diffuse and the specular source term on the right hand side of the equation. As a consequence, the model is quite sensitive to the specular part at singular points.

4. (NON-)CONVEXITY OF THE HAMILTONIAN

We complement the investigation of singular points with an analysis of the non-convexity of the Hamiltonian. The important issue we address by this analysis is that efficient numerical solvers like the iterative upwind technique [12] or the non-iterative FM method [8] are well-established just for the convex case. Therefore, applying the FM method as in [8], the question arises if one makes a serious error in the depth computation by neglecting the non-convex structure of the problem.

For simplicity, let us restrict ourselves to the 1-D case. The Hamiltonian H of the P^{2}SFS model is defined by writing the 1-D version of the HJE (1) in the format

$$H\,(x, v, v_x) = 0 \quad (13)$$

Substituting the variable p for v_x, our aim is to investigate the (non-)convexity of H with respect to p, i.e. we will investigate the sign of $\frac{\partial^2}{\partial p^2} H(x,v,p)$.

As it will turn out, the Hamiltonian will be convex in all parts with the possible exception of the contribution by the specular term. Since the positive factor $k_s I_s e^{-2v}$ within the specular term will simply be kept when differentiating w.r.t. p, we neglect it in the following. Thus, it suffices to consider the remaining part of the specular term

$$h_s\,(x,p) \;:=\; \frac{w(x)}{q(x)}\left(\frac{2q(x)^2}{w(x)^2} - 1\right)^\alpha \quad (14)$$

$$\text{with} \quad w(x) \;:=\; \sqrt{\mathsf{f}^2 p^2 + p^2 x^2 + q(x)^2} \quad (15)$$

$$\text{and} \quad q(x) \;:=\; \mathsf{f}/\left(\sqrt{x^2 + \mathsf{f}^2}\right) \quad (16)$$

Let us note that in H the term h_s is multiplied by -1, i.e. for convexity of H we need concavity of h_s. For simplicity, we consider now the abbreviations

$$A \;:=\; \frac{1}{q}\left(\frac{2q(x)^2}{w(x)^2} - 1\right)^{\alpha-1} \quad (17)$$

$$B \;:=\; p\frac{\mathsf{f}^2 + x^2}{w} \quad (18)$$

$$C \;:=\; \frac{2q(x)^2(1-2\alpha)}{w(x)^2} - 1 \quad (19)$$

Then one obtains after some computation

$$\frac{\partial^2}{\partial p^2} h_s = \frac{\partial A}{\partial p} BC + A\frac{\partial B}{\partial p} C + AB\frac{\partial C}{\partial p} \quad (20)$$

The search for a value p for which $\frac{\partial^2}{\partial p^2} h_s = 0$ gives a necessary condition for switching from convexity to concavity.

One can show, that $\frac{\partial^2}{\partial p^2} h_s = 0$ holds for $p = \pm \mathsf{f} / \left(x^2 + \mathsf{f}^2\right)$ which is a range of values where the pure (convex) Lambertian model dominates, as we know from the previous section.

Investigating singular points where $v_x = 0$, one obtains

$$\left. \frac{\partial^2}{\partial p^2} h_s \right|_{p=0} = \frac{\left(x^2 + \mathsf{f}^2\right)^2}{\mathsf{f}^2} (1 - 4\alpha) \tag{21}$$

i.e. convexity is guaranteed as long as $\alpha > 1/4$. Let us stress, that in practice one would usually set the value of α to 4 or higher.

In summary, the Hamiltonian of the P^2SFS model will be convex within the range of parameters interesting for practical computations. Therefore, efficient solvers like FM are well-defined for this model and work without a numerical problem.

5. NUMERICAL TEST

We demonstrate the importance of the specular term with a new synthetic experiment not contained in [7, 8], see Figure 2. Our test shows that the shape of the vase can be computed successfully even if eighty percent of the information arises by specular reflection at the singular points.

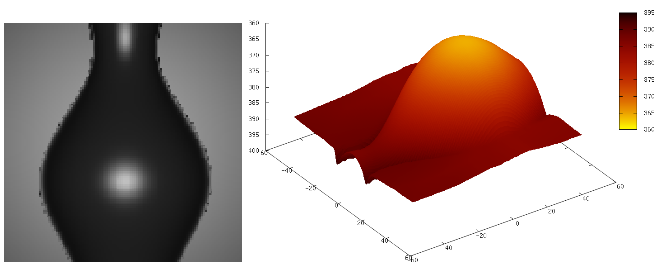

Fig. 2. P^2SFS for highly specular reflection, $k_a = 0$, $k_d = 0.2$, $k_s = 0.8$, $\alpha = 5$. *Left.* Original image. *Right.* 3-D reconstruction using the FM method. The average depth error of the reconstruction is around 2% of the true depth.

6. CONCLUSION

In this paper, we investigated an important prototype for non-Lambertian PSFS models. Our detailed analysis stresses the importance of the non-Lambertian part of the model and it has also revealed many mechanisms of P^2SFS. In addition, we have shown that the FM method works properly for the considered model. This may not be self-evident for other non-Lambertian model extensions.

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